ANALYTIC EXTENSION TECHNIQUES FOR UNITARY REPRESENTATIONS OF BANACH–LIE GROUPS

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Abstract. Let (G, θ) be a Banach–Lie group with involutive automorphism θ, $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ be the θ-eigenspaces in the Lie algebra $\mathfrak{g}$ of G, and $H = (G^\theta)_0$ be the identity component of its group of fixed points. An Olshanski semigroup is a semigroup $S \subseteq G$ of the form $S = H \exp(W)$, where W is an open $\mathrm{Ad}(H)$-invariant convex cone in $\mathfrak{q}$ and the polar map $H \times W \to S, (h, x) \mapsto h \exp x$ is a diffeomorphism. Any such semigroup carries an involution $*$ satisfying $(h \exp x)^* = (\exp x)h^{-1}$. Our central result, generalizing the Lüscher–Mack Theorem for finite dimensional groups, asserts that any locally bounded $*$-representation $\pi: S \to B(\mathcal{H})$ with a dense set of smooth vectors defines by "analytic continuation" a unitary representation of the simply connected Lie group G_c with Lie algebra $\mathfrak{g}_c = \mathfrak{h} + i\mathfrak{q}$. We also characterize those unitary representations of G_c obtained by this construction. With similar methods, we further show that semibounded unitary representations extend to holomorphic representations of complex Olshanski semigroups.

1. Introduction

There are many important results in the unitary representation theory of Lie groups related to analytic continuation. Here a key ingredient is the special case where $\pi: \mathbb{R} \to \mathrm{U}(\mathcal{H})$ is a strongly continuous unitary one-parameter group and A its self-adjoint infinitesimal generator, i.e., $\pi(t) = e^{itA}$ in the sense of measurable functional calculus. Then the unitary one-parameter group π extends to a holomorphic one-parameter semigroup $\widehat{\pi}: \mathbb{C}_+ = \mathbb{R} + i\mathbb{R}_{\geq 0} \to B(\mathcal{H})$ if and only if A is bounded below. This extension then restricts to a locally bounded non-degenerate hermitian one-parameter semigroup $t \mapsto \widehat{\pi}(it)$. Conversely, any such hermitian one-parameter group has a self-adjoint infinitesimal generator $-A$ and e^{izA} then yields an extension to $\mathbb{C}_+$, where the boundary values form a unitary one-parameter group. The key point of this picture is that self-adjoint operators A whose spectrum is bounded below can be viewed as the infinitesimal generators of two objects: a unitary one-parameter group (Stone's Theorem) and a hermitian one-parameter semigroup of bounded operators (Hille–Yosida Theorem). This is the one-parameter context of what we are dealing with in the present paper for Banach–Lie groups.

We call a pair (G, θ) consisting of a Banach–Lie group G and an involutive automorphism θ of G a *symmetric Banach–Lie group*. We decompose its Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ into ± 1-eigenspaces of θ and write $H := G_0^\theta$ for the identity component of the group of θ-fixed points in G. Let $W \subseteq \mathfrak{q}$ be an open convex $\mathrm{Ad}(H)$-invariant cone for which the polar map

$$H \times W \to H \exp(W), \quad (h, x) \mapsto h \exp x$$

is a diffeomorphism onto an open subsemigroup $S := S_H(W) := H \exp(W)$ of G. Then S is an involutive semigroup with respect to the involution $s^* := \theta(s)^{-1}$ which is called an *Olshanski semigroup*. In Appendix A we explain how to obtain such semigroups $S_{H'}(W)$, where H' is a connected Lie group locally isomorphic to H for which H' and $S_{H'}(W)$ need not be contained in a Lie group (cf. [Ne92, Ex. II.13]).

Our first main result (Theorem 3.7, proved in Section 4) is the following Banach version of the Lüscher–Mack Theorem ([LM75]): Let $\rho : S = S_H(W) \to B(\mathcal{H})$ be a non-degenerate strongly continuous $*$-representation of S which is *smooth* in the sense that the space $\mathcal{H}^\infty$ of smooth vectors is dense. If G_c is the simply connected Lie group with Lie algebra $\mathfrak{g}_c := \mathfrak{h} + i\mathfrak{q}$, then there exists a smooth unitary representation $(\pi, \mathcal{H})$ of G_c on $\mathcal{H}$ which is uniquely determined by the requirement that the unitary one-parameter groups corresponding to elements $x \in \mathfrak{h}$ are those obtained by "extending" ρ to H and the generators of the one-parameter groups corresponding to elements of $iW \subseteq i\mathfrak{q}$ are of the form $i\mathtt{d}\rho(x)$, where $\mathtt{d}\rho(x)$ is the infinitesimal generator of the hermitian one-parameter semigroup $t \mapsto \rho(\exp tx)$. For finite dimensional groups this result is due to M. Lüscher and G. Mack ([LM75]; see also [HN93]). As their methods make heavy use of coordinates obtained from products of one-parameter groups, we have to develop a completely new approach in the Banach context. Actually our approach is more direct and uses only quite general methods, such as the criteria for the integrability of infinitesimal unitary representations of Banach–Lie algebras from [Mer10]. The one-parameter case described above corresponds to $G = \mathbb{R}$, $\theta(g) = -g$, $S = \mathbb{R}_{>0}$ and $G_c = i\mathbb{R}$.

An interesting, much less involved, special case arises if $S = G$ is the whole group. Then our assumption is that G has a diffeomorphic polar decomposition $H \exp \mathfrak{q}$ and our theorem establishes a correspondence between $*$-representations $\pi\colon G \to \mathrm{GL}(\mathcal{H})$ by bounded operators and norm-continuous unitary representations of G_c. If G is a finite dimensional semisimple Lie group and θ a Cartan involution, then this is Weyl's well-known unitary trick relating finite dimensional representations of G to unitary representations of the compact group G_c.

In the context of the Lüscher–Mack Theorem, it is a natural question which unitary representations of G_c are obtained from representations of a semigroup $S_H(W)$. To answer this and related questions, we consider for a smooth unitary representation $(\pi, \mathcal{H})$ of a Lie group G the convex function

$$s_\pi\colon \mathfrak{g} \to \mathbb{R} \cup \{\infty\}, \quad s_\pi(x) := \sup\big(\mathrm{Spec}(i\mathtt{d}\pi(x))\big).$$

We call π *semibounded* if s_π is bounded in the neighborhood of some point $x_0 \in \mathfrak{g}$, and if $W \subseteq \mathfrak{g}$ is a convex cone, then we say that π is W*-semibounded* if s_π is locally bounded on W. Now the converse to the Lüscher–Mack Theorem (Corollary 5.6) asserts that a representation of G_c is obtained from a smooth representation of S if and only if it is iW-semibounded. Here the main difficulty is to show that, for an iW-semibounded unitary representation of G_c, the prescription $\rho(h \exp x) := \pi(h)e^{-\mathtt{d}\pi(x)}$ define a representation and to see that it is actually smooth. Here we use methods developed previously in [Mer10] and [Ne10b].

With the same tools we obtain the following Holomorphic Extension Theorem (cf. [Ol82] and [Ne00] for the finite dimensional case): For every open invariant cone $W \subseteq \mathfrak{g}$ and a corresponding complex Olshanski semigroup $S_G(iW)$, each W-semibounded unitary representation of G extends to a holomorphic representation of $S_G(iW)$.

In the finite dimensional context Lawson's Theorem guarantees the existence of Olshanski semigroups $S_H(W)$ under quite simple requirements on the spectra of the operators $\mathrm{ad}\, x$, $x \in W$ (cf. [La94], [Ne00]). Since Lawson's arguments involve local compactness in a crucial way, they do not generalize to the Banach context. However, natural examples of Olshanski semigroups arise as compression semigroups of bounded symmetric complex domains in Banach spaces, symmetric Hilbert domains, and real forms of such domains (cf. [Ne01]).

As we show in [MN11], the real forms $\mathcal{D}$ of symmetric Hilbert domains $\mathcal{D}_c$, i.e., the fixed points for an antiholomorphic involution σ, are of particular interest in

representation theory (see [Kau83, Kau97] for a classification). Here G_c is a central extension of the identity component of $\mathrm{Aut}(\mathcal{D}_c)$, which leads to the class of *hermitian Lie groups* whose semibounded representations are classified in [Ne10c]. Choosing a base point in $\mathcal{D}$ leads to an involution θ on G_c with $\mathcal{D}_c \cong G_c/G_c^\theta$, and conjugation with σ induces another involution on G_c. The complex domain $\mathcal{D}_c$ has a natural compression semigroup of form $S_c = G_c \exp(W_c)$, where $W_c \subseteq i\mathfrak{g}_c$ is an open invariant cone ([Ne01]). The subgroup $H := (G_c^\sigma)_0$ is invariant under θ and $\mathcal{D} \cong H/H^\theta$ is a real symmetric space. The corresponding semigroup is $S_H(W)$ for $W = W_c^\sigma$. Here our Lüscher–Mack Theorem provides a bridge between $*$-representations of $S_H(W)$ and unitary representations of the group G_c. A remarkable feature of the infinite dimensional context is that a large class of the irreducible separable continuous unitary representations of H extend to contraction representations of $S_H(W)$ with the same commutant and hence further to representations of the larger group G_c. In particular we obtain an automatic extension of a large class of irreducible representations of H to irreducible semibounded representations of G_c. For irreducible domains, the irreducible semibounded representations of G_c are classified in [Ne10c], and this in turn leads to a classification of the corresponding representations of H.

One of the central motivations to study results like the Lüscher–Mack Theorem is that there are natural sources of involutive representations of real Olshanski semigroups $S_H(W)$. Here the constructions based on "reflection positivity" are of particular interest because of their connections with euclidean, resp., relativistic quantum field theories (cf. [JOl00], [NO11], [LM75], [GJ81]). In this context the semigroup $S_H(W)$ constitutes a bridge between unitary representations of the groups G and G_c. However, these semigroups do not exist in all situations, where the passage from G to G_c is of interest, a typical example is the euclidean motion group $G = \mathbb{R}^4 \rtimes \mathrm{SO}_4(\mathbb{R})$ and the Poincaré group $G_c = \mathbb{R}^4 \rtimes \mathrm{SO}_{1,3}(\mathbb{R})$. This was the motivation for Fröhlich, Osterwalder and Seiler to introduce the concept of a virtual representations of a symmetric Lie group ([FOS83]). It would be very interesting to see if a suitable variant of this concept can be developed for Banach–Lie groups.

Notation and conventions. If $\mathcal{H}$ is a Hilbert space, we write $B(\mathcal{H})$ for the algebra of bounded linear operators on $\mathcal{H}$, $\mathrm{GL}(\mathcal{H})$ for its group of units, and $\mathrm{U}(\mathcal{H}) \subseteq \mathrm{GL}(\mathcal{H})$ for the unitary group.

If G is a topological group and $\mathcal{H}$ a complex Hilbert space, then a *unitary representation of G on $\mathcal{H}$*, denoted $(\pi, \mathcal{H})$, is a homomorphism $\pi \colon G \to \mathrm{U}(\mathcal{H})$ which is continuous with respect to the strong operator topology, i.e., all orbit maps $\pi^v \colon G \to \mathcal{H}, g \mapsto \pi(g)v$ are continuous.

If G is a Banach–Lie group, then we write $\mathfrak{g}$ for its Lie algebra. It is a Banach–Lie algebra, i.e., a Banach space with a continuous Lie bracket.

If S is a semigroup, we write $\lambda_s(t) = st$ and $\rho_s(t) = ts$ for left and right multiplications on S. If a neutral element in S exists, it is denoted by e.

Let M be a set. A kernel function $K \colon M \times M \to \mathbb{C}$ is called *positive definite* if for $x_1, \ldots, x_n \in M$, $n \in \mathbb{N}$, the matrix $(K(x_i, x_j))_{i,j=1,\ldots n}$ is positive definite. For any such kernel there exists a unique Hilbert subspace $\mathcal{H}_K \subseteq \mathbb{C}^M$ of functions on M with continuous evaluation maps given by $f(m) = \langle f, K_m \rangle$, $K_m(x) = K(x, m), x, m \in M$. In particular, the subspace $\mathcal{H}_K^0$, spanned by the functions K_m, $m \in M$, is dense in $\mathcal{H}_K$. If $\gamma \colon M \to \mathcal{H}$ is a function with values in a Hilbert space whose range is total in the sense that it spans a dense subspace and $\langle \gamma(x), \gamma(y) \rangle = K(y, x)$ for $x, y \in M$, then we have a unitary map $\Phi_\gamma \colon \mathcal{H} \to \mathcal{H}_K, \Phi_\gamma(v)(x) = \langle v, \gamma(x) \rangle$ (cf. [Ne00, Thm. I.1.6]).

2. Geometric symmetric operators on reproducing kernel spaces

Let $\mathcal{M}$ be a Banach manifold, $K \in C^\infty(\mathcal{M}\times\mathcal{M},\mathbb{C})$ be a smooth positive definite kernel on $\mathcal{M}$ and $\mathcal{H}_K \subseteq \mathbb{C}^M$ be the corresponding reproducing kernel Hilbert space. According to [Ne10b, Thm. 7.1], the map $\mathcal{M} \to \mathcal{H}_K, m \mapsto K_m$, is smooth, so that $\varphi(m) = \langle \varphi, K_m\rangle$ for $\varphi \in \mathcal{H}_K$ implies that $\mathcal{H}_K \subseteq C^\infty(\mathcal{M},\mathbb{C})$.

Let V be a vector field on $\mathcal{M}$, $\mathcal{L}_V$ the associated derivation of $C^\infty(\mathcal{M},\mathbb{C})$, and φ_t^V be its local flow, defined at $m \in \mathcal{M}$ for $0 \leq t < \epsilon(m)$. The Lie derivative on functions is given by

$$(\mathcal{L}_V\varphi)(m) := \frac{d}{dt}\Big|_{t=0}\varphi(\varphi_t^V m). \tag{1}$$

We also consider $\mathcal{L}_V$ as an unbounded operator on $\mathcal{H}_K$ defined on the domain

$$\mathcal{D}_V := \{\varphi \in \mathcal{H}_K \mid \mathcal{L}_V\varphi \in \mathcal{H}_K\}.$$

Definition 2.1. We say that V is *symmetric with respect to* K (or K*-symmetric*), if

$$(\mathcal{L}_V K_m)(n) = \overline{(\mathcal{L}_V K_n)(m)} \quad \text{for} \quad m, n \in \mathcal{M}.$$

The key point in the preceding definition is that it can be expressed in terms of the kernel and the vector field, but the following proposition shows that we can draw interesting conclusions for the corresponding unbounded operator on $\mathcal{H}_K$.

Proposition 2.2. *Let V be a K-symmetric vector field. Then, for every $m \in \mathcal{M}$,*

$$\mathcal{L}_V K_m = \frac{d}{dt}\Big|_{t=0} K_{\varphi_t^V m} \in \mathcal{H}. \tag{2}$$

In particular $\mathcal{H}_K^0 \subseteq \mathcal{D}_V$ and the operator $\mathcal{L}_V|_{\mathcal{H}_K^0}$ is symmetric. If $\varphi \in \mathcal{H}$ then

$$\langle \varphi, \mathcal{L}_V K_m\rangle = (\mathcal{L}_V\varphi)(m) \quad \textit{for} \quad m \in \mathcal{M}. \tag{3}$$

Proof. Since $\mathcal{M} \to \mathcal{H}_K$, $m \to K_m$ is smooth, the derivative $\frac{d}{dt}K_{\varphi_t^V m}$ exists in $\mathcal{H}_K$ and, for every $n \in \mathcal{M}$,

$$(\mathcal{L}_V K_m)(n) = \overline{(\mathcal{L}_V K_n)(m)} = \frac{d}{dt}\Big|_{t=0} K(n, \varphi_t^V m) = \Big(\frac{d}{dt}\Big|_{t=0} K_{\varphi_t^V m}\Big)(n).$$

For every $\varphi \in \mathcal{H}$ we therefore have

$$\langle \varphi, \mathcal{L}_V K_m\rangle = \langle \varphi, \frac{d}{dt}\Big|_{t=0} K_{\varphi_t^V m}\rangle = \frac{d}{dt}\Big|_{t=0}\varphi(\varphi_t^V m) = (\mathcal{L}_V\varphi)(m). \qquad \square$$

Remark 2.3. Note that (3) implies in particular

$$\langle \varphi, \mathcal{L}_V K_m\rangle = \langle \mathcal{L}_V\varphi, K_m\rangle \quad \text{for} \quad \varphi \in \mathcal{D}_V. \tag{4}$$

Below we recall Fröhlich's Theorem on unbounded symmetric semigroups as it is stated in [Frö80, Cor. 1.2]. Actually Fröhlich assumes that the Hilbert space $\mathcal{H}$ is separable, but this is not necessary for the conclusion. Replacing the assumption of weak measurability by weak continuity, all arguments in [Frö80] work for non-separable spaces as well.

Theorem 2.4 (Fröhlich)**.** *Let H be a symmetric operator defined on the domain $\mathcal{D}$ dense in the Hilbert space $\mathcal{H}$. Suppose that for every $\Phi \in \mathcal{D}$ there exists $\epsilon(\Phi) > 0$ such that the equation*

$$\frac{d}{dt}\Phi(t) = H\Phi(t)$$

has a solution satisfying $\lim_{t\to 0}\Phi(t) = \Phi$ *and $\Phi(t) \in \mathcal{D}$ for $0 \leq t < \epsilon(\Phi)$. Then the operator H is essentially self-adjoint and $\Phi(t) = e^{t\overline{H}}\Phi$.*

We apply Fröhlich's Theorem to our geometric operator $\mathcal{L}_V$ from Proposition 2.2. The new feature is that we can even describe the closure of the essentially self-adjoint operator we obtain.

Theorem 2.5 (Geometric Fröhlich Theorem). *Let $\mathcal{M}$ be a Banach manifold and K be a smooth positive definite kernel. Then $\mathcal{H}_K$ consists of smooth functions, and if V is a K-symmetric vector field on $\mathcal{M}$, then the Lie derivative $\mathcal{L}_V$ defines an essentially self-adjoint operator $\mathcal{H}_K^0 \to \mathcal{H}_K$ whose closure $\mathcal{L}_V^K$ coincides with $\mathcal{L}_V|_{\mathcal{D}_V}$. Moreover, if the local flow $\Phi_t^V(m)$ of $m \in \mathcal{M}$ at time t is defined, then*

$$e^{t\mathcal{L}_V^K} K_m = K_{\Phi_t^V(m)}.$$

Proof. We apply Fröhlich's Theorem with $\mathcal{D} = \mathcal{H}_K^0$ and $H = \mathcal{L}_V|_{\mathcal{D}}$. For $\Phi = \sum_{j=1}^k \alpha_j K_{m_j}$, $\alpha_j \in \mathbb{C}$, $m_j \in M$, we define

$$\Phi(t) = \sum_{j=1}^k \alpha_j K_{\varphi_t^V m_j}, \qquad \text{for} \qquad t < \min(\epsilon(m_1), \ldots, \epsilon(m_k)).$$

Then (2) implies $\frac{d}{dt}\Phi(t) = \mathcal{L}_V \Phi(t)$, so the assumptions of Fröhlich's Theorem are satisfied, and $\mathcal{L}_V|_{\mathcal{H}_K^0}$ is essentially self-adjoint. Let us denote by T_V its closure. If φ is an element of its domain $\mathcal{D}(T_V)$, then (3) leads for every $m \in \mathcal{M}$ to

$$(T_V\varphi)(m) = \langle T_V\varphi, K_m\rangle = \langle \varphi, \mathcal{L}_V K_m\rangle = (\mathcal{L}_V\varphi)(m).$$

We conclude that $\mathcal{D}(T_V) \subseteq \mathcal{D}_V$ and $T_V = \mathcal{L}_V|_{\mathcal{D}(T_V)}$. The other inclusion follows from (4) which implies that $\mathcal{D}_V \subseteq \mathcal{D}((\mathcal{L}_V|_{\mathcal{H}_K^0})^*)$. Indeed, since $\mathcal{L}_V|_{\mathcal{H}_K^0}$ is essentially self-adjoint we have $(\mathcal{L}_V|_{\mathcal{H}_K^0})^* = T_V$. □

Remark 2.6. Theorem 2.4 above extends directly to the case where $\mathcal{M}$ is a manifold modeled on a locally convex space, provided the vector field V is assumed to have a local flow. Indeed [Ne10b, Thm. 7.1] is stated in this generality.

3. The Lüscher–Mack Theorem

Let (G, θ) be a symmetric Banach–Lie group. We also write θ for the corresponding automorphism of its Lie algebra $\mathfrak{g}$, which leads to the *symmetric Banach–Lie algebra* $(\mathfrak{g}, \theta)$. We write

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q} \qquad \text{with} \qquad \mathfrak{h} = \ker(\theta - \mathbf{1}) \qquad \text{and} \qquad \mathfrak{q} = \ker(\theta + \mathbf{1}),$$

for the eigenspace decomposition of $\mathfrak{g}$ under θ. Let $\emptyset \neq W \subseteq \mathfrak{q}$ be an open convex cone invariant under $e^{\mathrm{ad}\,\mathfrak{h}}$ and $S = S_H(W)$ be a corresponding Olshanski semigroup in the sense of Definition A.4. In particular, H is a connected Lie group with Lie algebra $\mathfrak{h}$, but we do not assume that H is contained in G. We write

$$\exp\colon W \to S = S_H(W)$$

for the corresponding exponential map on W. In polar coordinates the involution on S is given by

$$(h \exp x)^* = (\exp x)h^{-1} = h^{-1}\exp(\mathrm{Ad}(h)x) \tag{5}$$

(Remark A.5).

Examples 3.1. (a) Let $\mathcal{H}$ be a real Hilbert space and $G = \mathrm{GL}(\mathcal{H})_0$. We write $A^\top$ for the adjoint of an element $A \in B(\mathcal{H})$. Then $\theta(g) = (g^\top)^{-1}$ defines an involution on G with $H = G^\theta = \mathrm{O}(\mathcal{H})_0$ (the orthogonal group of $\mathcal{H}$) and $\mathfrak{q} = \mathrm{Sym}(\mathcal{H})$ is the space of symmetric operators. In this case $G = \mathrm{O}(\mathcal{H})_0 \exp(\mathrm{Sym}(\mathcal{H}))$ actually is an Olshanski (semi-)group for $W = \mathfrak{q} = \mathrm{Sym}(\mathcal{H})$. Similarly statements hold for complex and quaternionic Hilbert spaces.

(b) If H is a connected Lie group for which $\mathrm{Ad}(H)$ leaves a compatible norm on $\mathfrak{h}$ invariant, then $\mathfrak{h}$ is called an *elliptic Banach–Lie algebra*. Then H has a universal complexification $\eta\colon H \to H_{\mathbb{C}}$ with a polar decomposition $H_{\mathbb{C}} = H\exp(i\mathfrak{h})$ ([Ne02, Ex. 6.9]). A finite dimensional Lie algebra $\mathfrak{h}$ is elliptic if and only if it is compact, but the class of elliptic Lie algebras is quite large. In particular, it contains the algebra $\mathfrak{u}(\mathcal{A})$ of skew-hermitian elements of a C^*-algebra $\mathcal{A}$ and in particular the Lie algebra $\mathfrak{u}(\mathcal{H})$ of the full unitary group $\mathrm{U}(\mathcal{H})$ of a complex Hilbert space $\mathcal{H}$.

(c) If V is a Banach space and $W \subseteq V$ an open convex cone, then $S = S_V(W) = V + iW \subseteq V_{\mathbb{C}}$ is a complex Olshanski semigroup with respect to the involution $(x+iy)^* := -x + iy$.

(d) If $\mathcal{A}$ is a unital C^*-algebra, $G := \mathcal{A}_0^\times$ (the identity component of its group of units) and $S := \{s \in G\colon \|s\| < 1\}$, then S is a complex Olshanski semigroup $S = S_{\mathrm{U}(\mathcal{A})_0}(W)$, where

$$W = \{x \in \mathcal{A}\colon x^* = x, \sup(\mathrm{Spec}(x)) < 0\}.$$

An important example is the semigroup of invertible strict contractions of a complex Hilbert space $\mathcal{H}$.

(e) Let $\mathcal{A}$ be a unital C^*-algebra and $\tau = \tau^2 = \tau^* \in \mathcal{A}$. For $a, b \in \mathcal{A}$, we write $a < b$ if there exists an invertible element $c \in \mathcal{A}$ with $b - a = c^*c$. Then

$$S := \{s \in \mathcal{A}^\times : s^*\tau s < \tau\}$$

is an open subsemigroup of $\mathcal{A}$ with respect to multiplication. To see that it is non-empty, we observe that we may write $\tau = \mathbf{1} - 2p = (\mathbf{1} - p) - p$ for a projection $p = p^* = p^2 \in \mathcal{A}$. For $\lambda \in \mathbb{C}^\times$ and $s := \lambda(\mathbf{1} - p) + \lambda^{-1}p$ we then have

$$s^*\tau s = |\lambda|^2(\mathbf{1} - p) - |\lambda^{-1}|^2 p < \tau = (\mathbf{1} - p) - p$$

if and only if $|\lambda| < 1$. The semigroup S is of the form $S_H(W)$ for

$$H = \{g \in \mathcal{A}^\times : g^*\tau g = \tau\}.$$

(f) As already mentioned in the introduction, the compression semigroups of (real forms of) symmetric (Hilbert) domains are also Olshanski semigroups (cf. [Ne01]).

Definition 3.2. Let $(S, *)$ be an *involutive Banach semigroup*, i.e., an involutive semigroup carrying a Banach manifold structure such that multiplication and inversion are smooth maps.

(a) A homomorphism $\rho\colon S \to B(\mathcal{H})$ is called a *$*$-representation* if $\rho(s^*) = \rho(s)^*$ for every $s \in S$. Such a representation is said to be *non-degenerate* if $\rho(S)\mathcal{H}$ spans a dense subspace of $\mathcal{H}$, which is equivalent to the condition that $\rho(S)v = \{0\}$ implies $v = 0$.

(b) For a representation $(\rho, \mathcal{H})$ of S, a vector $v \in \mathcal{H}$ is called *smooth* if its orbit map $\rho^v\colon S \to \mathcal{H}, s \mapsto \rho(s)v$ is smooth. We write $\mathcal{H}^\infty$ for the subspace of *smooth vectors* and say that $(\rho, \mathcal{H})$ is *smooth* if $\mathcal{H}^\infty$ is dense in $\mathcal{H}$.

(c) A $*$-representation $(\rho, \mathcal{H})$ of S is called *locally bounded* if every $s \in S$ has a neighborhood on which $\|\rho(\cdot)\|$ is bounded.

(d) For a unitary representation $(\pi, \mathcal{H})$ of a Lie group G, we also write $\mathcal{H}^\infty := \mathcal{H}^\infty(\pi)$ for the subspace of smooth vectors. This subspace carries the *derived representation*

$$\mathtt{d}\pi\colon \mathfrak{g} \to \mathrm{End}(\mathcal{H}^\infty), \quad \mathtt{d}\pi(x)v := \frac{d}{dt}\Big|_{t=0} \pi(\exp tx)v$$

of the Lie algebra $\mathfrak{g}$ of G. If π is smooth, then these operators are essentially skew-adjoint and, for $x \in \mathfrak{g}$, the closure $\overline{\mathtt{d}\pi}(x)$ is the infinitesimal generator of the unitary one-parameter group $\pi(\exp tx)$.

Remark 3.3. (a) Since the arguments in [Ne10b, Prop. 5.1, Lemma 5.2] apply also to semigroup actions, the first countability of S implies that any strongly continuous representation $\rho\colon S \to B(\mathcal{H})$ defines a continuous action $S \times \mathcal{H} \to \mathcal{H}$. The continuity in the points $(s, 0)$ implies that ρ is locally bounded.

(b) If, conversely, ρ is locally bounded and the orbit maps $\rho^v\colon S \to \mathcal{H}$ are continuous for every v in a dense subspace, then ρ is strongly continuous ([Ne00, Lemma IV.1.3]).

In the following we are interested in non-degenerate strongly continuous $*$-representations $\rho : S = S_H(W) \to B(\mathcal{H})$ on a Hilbert space $\mathcal{H}$. First we observe that, although H need not be contained in S, any non-degenerate $*$-representation of S defines in a natural fashion a unitary representation of H.

Proposition 3.4. *For every non-degenerate $*$-representation $(\rho, \mathcal{H})$ of S there exists a unique, not necessarily continuous, unitary representation $\rho_H\colon H \to \mathrm{U}(\mathcal{H})$ satisfying*

$$\rho(hs) = \rho_H(h)\rho(s) \quad \text{for} \quad h \in H, s \in S. \tag{6}$$

Its space of continuous vectors contains $\rho(S)\mathcal{H}$ and its space of smooth vectors contains $\rho(S)\mathcal{H}^\infty$. In particular, $(\rho_H, \mathcal{H})$ is strongly continuous, resp., smooth if $(\rho, \mathcal{H})$ is.

Proof. The existence of a unique homomorphism $\rho_H\colon H \to \mathrm{U}(\mathcal{H})$ satisfying (6) follows from the fact that H acts on S by unitary multipliers (Remark A.5, [Ne00, Rem. III.1.5]). The smoothness of the action of H on S now implies that $\rho(s)v$ has a continuous (smooth) orbit map under H if v has a continuous (smooth) orbit map under S. This completes the proof. □

To apply the Hille–Yosida Theorem to the symmetric one-parameter semigroups $\rho_x(t) := \rho(\exp tx)$ for $x \in W$, we need to know that

$$\lim_{t\to 0} \rho_x(t)v = v \text{ for } v \in \mathcal{H}. \tag{7}$$

As the following lemma shows, this follows from the non-degeneracy of the representation and strong continuity.

Lemma 3.5. *If ρ is a non-degenerate $*$-representation of $S = S_H(W)$ and $x \in W$, then $\rho(\exp x)\mathcal{H}$ is dense in $\mathcal{H}$. Furthermore, (7) holds if ρ_x is strongly continuous on $\mathbb{R}_{>0}$.*

Proof. Let $v \in (\rho(\exp x)\mathcal{H})^\perp$. In view of

$$\langle \rho(\exp x)w, v\rangle = \langle w, \rho(\exp x)v\rangle,$$

this is equivalent to $\rho(\exp x)v = 0$. Now [Ne00, Cor. II.4.15] implies that

$$\rho\Big(\exp\frac{x}{n}\Big)v = 0 \quad \text{for every} \quad n > 0$$

and hence that $\rho(\exp tx)v = 0$ for every $t > 0$.

Any $s \in S$ can be written $s = s_0 \exp(tx)$ with $s_0 \in S$ and some $t > 0$ because the left invariant vector field V_x generates a local flow on S (Remark A.9). It follows that $\rho(s)v = \rho(s_0)\rho(\exp tx)v = 0$. Since ρ is a non-degenerate representation of S, it follows that $v = 0$, and hence that $\rho(\exp x)\mathcal{H}$ is dense in $\mathcal{H}$.

We see in particular, that the representation ρ_x of $\mathbb{R}_{>0}$ is non-degenerate. Assume that ρ_x is strongly continuous, hence locally bounded (Remark 3.3). Then Lemma [Ne00, Lemma VI.2.2] implies that ρ_x extends uniquely to a strongly continuous representation on $\mathbb{R}_{\geq 0}$, which implies (7). □

Definition 3.6. Let $(\rho, \mathcal{H})$ be a non-degenerate strongly continuous $*$-representation of S. In view of Lemma 3.5, we obtain for every $x \in W$ a self-adjoint operator

$$\overline{\mathrm{d}\rho}(x)\xi := \frac{d}{dt}\Big|_{t=0} \rho(\exp tx)\xi,$$

the generator of the symmetric one-parameter semigroup ρ_x. It is defined on the subspace $\mathcal{D}(\overline{\mathrm{d}\rho}(x))$ where the derivative exists (Hille–Yosida Theorem). For $x \in \mathfrak{h}$, we likewise write $\overline{\mathrm{d}\rho}(x)$ for the generator of the corresponding strongly continuous unitary one-parameter group $\rho_x(t) := \rho_H(\exp tx)$ (Proposition 3.4; Stone's Theorem).

The following theorem is our main result.

Theorem 3.7. *Let G_c be a simply connected Lie group with Lie algebra*

$$\mathfrak{g}_c = \mathfrak{h} \oplus i\mathfrak{q} \subseteq \mathfrak{g}_{\mathbb{C}},$$

and $\rho : S = S_H(W) \to B(\mathcal{H})$ be a non-degenerate strongly continuous smooth $*$-representation. Then there exists a unique smooth unitary representation $(\pi, \mathcal{H})$ of G_c on $\mathcal{H}$ whose space of smooth vectors is contained in $\mathcal{D}(\overline{\mathrm{d}\rho}(x))$ for every $x \in \mathfrak{h} \cup W$, and whose derived representation satisfies

$$\mathrm{d}\pi(x + iy) \subseteq \overline{\mathrm{d}\rho}(x) + i\,\overline{\mathrm{d}\rho}(y) \quad \textit{for } x \in \mathfrak{h} \textit{ and } y \in W. \tag{8}$$

We will see (Remark 4.10) that the space $\mathcal{H}^\infty(\pi)$ of smooth vectors for the representation π of G_c coincides with

$$\bigcap_{x_j \in \mathfrak{h} \cup W,\ n \in \mathbb{N}} \mathcal{D}\big(\overline{\mathrm{d}\rho}(x_n)) \ldots \overline{\mathrm{d}\rho}(x_1)\big).$$

Accordingly, our strategy is to define a representation of the Lie algebra $\mathfrak{g}_c$ on this space so that (8) is satisfies and then verify that we can use the results in [Mer10] to show that this representation of $\mathfrak{g}_c$ integrates to a representation of G_c.

Remark 3.8. (a) Let $q_H : \widetilde{H} \to H$ be the simply connected covering group of H. Then we have a unique morphism $\iota : \widetilde{H} \to G_c$ integrating the inclusion map $\mathfrak{h} \to \mathfrak{g}_c$. The relation $\pi \circ \iota = \rho_H \circ q_H$ now implies that $\iota(\ker q_H) \subseteq \ker \pi$. As $\ker q_H$ acts trivially on $\mathfrak{g}$, it also acts trivially on $\mathfrak{g}_c$, i.e., $\iota(\ker q_H) \subseteq Z(G_c)$. In particular, it is a normal subgroup and π actually factors through a representation of the quotient $G_c/\iota(\ker q_H)$. If $\iota(\ker q_H)$ is discrete, this quotient is a Lie group with the same Lie algebra.

Here is an example showing that the group $\iota(\ker q_H)$ need not be discrete. We consider the Lie algebra $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R}) \oplus \mathbb{R}$ with the involution $\theta(x, t) = (-x^\top, t)$, which leads to $\mathfrak{g}_c = \mathfrak{su}_2(\mathbb{C}) \oplus \mathbb{R}$, and $G_c := \mathrm{SU}_2(\mathbb{C}) \times \mathbb{R}$ is a corresponding simply connected group.

For the Lie algebra $\mathfrak{h} = \mathfrak{g}^\theta = \mathfrak{so}_2(\mathbb{R}) \oplus \mathbb{R}$, the corresponding simply connected group is $\widetilde{H} = \mathbb{R}^2$, and the kernel for the adjoint action of $\widetilde{H}$ on $\mathfrak{g}$ is isomorphic to $\mathbb{Z} \times \mathbb{R}$ with $\ker \iota = 2\mathbb{Z} \times \{0\}$. Now $\Gamma := \mathbb{Z}(2, 1) \oplus \mathbb{Z}(0, \sqrt{2})$ is a discrete subgroup of $\widetilde{H}$ acting trivially on $\mathfrak{g}$, so that $H := \widetilde{H}/\Gamma$ satisfies our condition imposed for the construction of Olshanski semigroups and $\ker q_H = \Gamma$. Now $\iota(\Gamma) = \{\mathbf{1}\} \times (\mathbb{Z} + \sqrt{2}\mathbb{Z})$ is not discrete in G_c.

(b) Let $q_S : \widetilde{S} = S_{\widetilde{H}}(W)$ be the universal covering of the Olshanski semigroup $S = S_H(W)$ and $(\rho, \mathcal{H})$ be a smooth locally bounded $*$-representation of S, so that Theorem 3.7 leads to a unitary representation $(\pi, \mathcal{H})$ of G_c. Clearly, the representation $\widetilde{\rho} := \rho \circ q_S$ of $\widetilde{S}$ leads to the same representation of G_c.

Now $\pi \circ \iota = \widetilde{\rho}_H$ implies that $\ker(\widetilde{\rho}_H) \supseteq \ker \iota$, so that the representation $\widetilde{\rho}$ of $\widetilde{S}$ actually factors through a representation of the semigroup $S_c := S_{H_c}(W)$, where $H_c = \iota(\widetilde{H})$. From the point of view of the representation theory of the group G_c, all

representations obtained by Theorem 3.7 can also be obtained from the semigroup S_c, which is locally isomorphic to S.

The representation $(\pi, \mathcal{H})$ of G_c obtained from the Lüscher–Mack Theorem has the remarkable property that the spectrum of the operator $i\overline{d\pi}(x)$ is bounded from below for every x in the cone $iW \subseteq \mathfrak{g}_c$. In Section 5 we will prove a converse to Theorem 3.7: A $-iW$-semibounded representation π of G_c comes from a strongly continuous smooth $*$-representation of $S_H(W)$, where $H = \langle \exp_{G_c} \mathfrak{h}\rangle$ is the identity component of the group of fixed points for the involution θ_c on G_c acting on the Lie algebra by $x + iy \mapsto x - iy$ for $x \in \mathfrak{h}, y \in \mathfrak{q}$.

Remark 3.9. It is instructive to take a closer look at the special case $\mathfrak{g} = \mathfrak{h}_{\mathbb{C}}$ with $\mathfrak{q} = i\mathfrak{h}$, i.e., where $\theta(z) = \overline{z}$ is complex conjugation with respect to the real form $\mathfrak{h}$. Then $S_H(W)$ is a complex Olshanski semigroup (cf. Definition A.4(b)). Then the complexification of $\mathfrak{g}$ can be realized by the embedding

$$\eta\colon \mathfrak{g} \to \mathfrak{g} \oplus \mathfrak{g}, \quad z \mapsto (z, \overline{z}),$$

which leads to $\mathfrak{g}_{\mathbb{C}} \cong \mathfrak{g} \oplus \mathfrak{g}$, a direct sum of two complex Lie algebras. In this picture the Lie algebra $\mathfrak{g}_c = \mathfrak{h} + i\mathfrak{q}$ is given by

$$\{(x,x)\colon x \in \mathfrak{h}\} + i\{(y,-y)\colon y \in i\mathfrak{h}\} = \{(x,x)\colon x \in \mathfrak{h}\} + \{(y,-y)\colon y \in \mathfrak{h}\} = \mathfrak{h} \oplus \mathfrak{h}.$$

We conclude that $G_c \cong \widetilde{H} \times \widetilde{H}$, where $\widetilde{H}$ is the simply connected covering group of H.

If $\alpha\colon \mathfrak{g} \to \mathrm{End}(\mathcal{D})$ is a complex linear representation of $\mathfrak{g}$ on the complex linear space $\mathcal{D}$, then the complex linear extension to $\mathfrak{g}_{\mathbb{C}} \cong \mathfrak{g} \oplus \mathfrak{g}$ is given by

$$\alpha_{\mathbb{C}}(z,w) = \tfrac{1}{2}\big(\alpha(z + \overline{w}) + i\alpha(i\overline{w} - iz)\big) = \alpha(z)$$

because

$$(z,w) = \tfrac{1}{2}\big((z + \overline{w}, \overline{z} + w) + i(i\overline{w} - iz, \overline{i\overline{w} - iz})\big).$$

Therefore the corresponding unitary representation π_c of G_c is given by $\pi_c(h_1, h_2) = \rho_H(h_1)$.

4. Proof of the Lüscher–Mack Theorem

The following lemma permits us to reduce the proof of Theorem 3.7 to the case where $(\rho, \mathcal{H})$ is cyclic and generated by a smooth vector.

Lemma 4.1. *A non-degenerate strongly continuous smooth $*$-representation is a direct sum of cyclic representations with smooth cyclic vectors.*

Proof. The set of all families $(\mathcal{H}_j)_{j\in J}$ of mutually orthogonal closed S-invariant subspaces which contain a smooth cyclic vector is well ordered by inclusion and by Zorn's Lemma it has a maximal element $(\mathcal{H}_j)_{j\in J}$. Let $\mathcal{K} = \overline{\bigoplus_{j\in J} \mathcal{H}_j}$. Then $\mathcal{K}$ and $\mathcal{K}^\perp$ are S-invariant, and we claim that if $\mathcal{K}^\perp$ is non-zero, it contains a smooth vector. Indeed, if $\mathcal{K} \neq \mathcal{H}$, there exists a smooth vector w which is not in $\mathcal{K}$. Let us write pr for the orthogonal projection on $\mathcal{K}^\perp$ and $v := \mathrm{pr}(w)$. Then the relation $\rho(s)v = \mathrm{pr}(\rho(s)w)$, $s \in S$, shows that v is a smooth vector in $\mathcal{K}^\perp$. Since ρ is non-degenerate, we have $v \in \overline{\rho(S)v}$ (cf. [Ne00, Lemma II.2.4]), hence $\overline{\mathrm{span}\,\rho(S)v}$ is a closed S-invariant subspace with a smooth cyclic vector orthogonal to each $\mathcal{H}_j$, $j \in J$. But this contradicts the maximality of $(\mathcal{H}_j)_{j\in J}$. Therefore $\mathcal{H} = \mathcal{K}$, and as representations of S, we have $\mathcal{H} \simeq \widehat{\bigoplus}_{j\in J} \mathcal{H}_j$. ☐

From now on we assume that $v_0 \in \mathcal{H}^\infty$ is a cyclic vector. Then we obtain a smooth positive definite kernel

$$K(s_1, s_2) := K_{s_2}(s_1) := \langle \rho(s_1 s_2^*) v_0, v_0\rangle$$

on S which leads to a unitary map

$$\Psi\colon \mathcal{H} \to \mathcal{H}_K, \quad \Psi(v)(s) := \rho^{v,v_0}(s) = \langle \rho(s)v, v_0\rangle = \langle v, \rho(s^*)v_0\rangle$$

onto the reproducing kernel space $\mathcal{H}_K \subseteq C^\infty(S,\mathbb{C})$. Indeed, Ψ intertwines ρ with the representation of S on $\mathcal{H}_K \subseteq C^\infty(S,\mathbb{C})$ given by $(\rho(s_2)\varphi)(s_1) = \varphi(s_1 s_2)$. Therefore it suffices to prove Theorem 3.7 only for the representation $(\rho, \mathcal{H}_K)$ of $S = S_H(W)$.

The function $\varphi_0 = \rho^{v_0,v_0} = \Psi(v_0)$ satisfies $\rho(s)\varphi_0 = K_{s^*}$, so that

$$\mathcal{H}_K^0 = \operatorname{span}(\rho(S)\varphi_0) \subseteq \mathcal{H}_K^\infty \quad \text{and} \quad \rho(s_2)K_{s_1} = K_{s_1 s_2^*} \tag{9}$$

for $s_1 \in S, s_2 \in S \cup H$. To each $x \in \mathfrak{g}$ we associate the corresponding left invariant vector field V_x on S (cf. Definition A.8) and write the corresponding integral curves as $s\exp(tx)$ (cf. Remark A.9). As in (1), we set

$$\mathcal{L}_x := \mathcal{L}_{V_x} \quad \text{and} \quad \mathcal{L}_x^K := \mathcal{L}_x|_{\mathcal{D}_x} \quad \text{for } \mathcal{D}_x := \mathcal{D}_{V_x} = \{\varphi \in \mathcal{H}_K \colon \mathcal{L}_x\varphi \in \mathcal{H}_K\},$$

and extend this definition $\mathbb{C}$-linearly to any $x \in \mathfrak{g}_\mathbb{C}$. Then we have on $C^\infty(S,\mathbb{C})$ the relation

$$[\mathcal{L}_x, \mathcal{L}_y] = \mathcal{L}_{[x,y]}. \tag{10}$$

We recall that a *core* of a self-adjoint operator $A\colon \mathcal{D} \to \mathcal{H}$ is a dense subspace $\mathcal{D}_0 \subseteq \mathcal{D}$ for which $A = \overline{A|_{\mathcal{D}_0}}$.

Proposition 4.2. *Let $x \in \mathfrak{h} \cup W$. Then $\mathcal{H}_K^0 \subseteq \mathcal{D}(\overline{\mathrm{d}\rho}(x))$, $\mathcal{H}_K^0$ is a core for $\overline{\mathrm{d}\rho}(x)$, and*

$$\overline{\mathrm{d}\rho}(x) = \mathcal{L}_x^K. \tag{11}$$

Proof. For $x \in \mathfrak{h}$, the first assertion follows from (9) and Proposition 3.4. For $x \in W$ and $s \in S$, the map $t \mapsto \rho(\exp(tx)s)\varphi_0$ extends to a smooth map on some 0-neighborhood, and for $t \geq 0$ it coincides with $e^{t\overline{\mathrm{d}\rho}(x)}\rho(s)\varphi_0$. Therefore $\rho(s)\varphi_0$ is contained in the domain $\mathcal{D}(\overline{\mathtt{d}\rho}(x))$ of $\overline{\mathtt{d}\rho}(x)$.

In the following we also write $\rho(h) := \rho_H(h)$ for $h \in H$. For $x \in \mathfrak{h} \cup W$, the space $\mathcal{H}_K^0$ is invariant under $\rho(\exp tx)$, $t > 0$, hence is a core for $\overline{\mathrm{d}\rho}(x)$ ([EN00, Prop. 1.7]), i.e., $\overline{\mathtt{d}\rho}(x) = \overline{\overline{\mathtt{d}\rho}(x)|_{\mathcal{H}_K^0}}$.

Finally, let $x \in \mathfrak{h} \cup W$ and $\varphi \in \mathcal{D}(\overline{\mathrm{d}\rho}(x))$. Then

$$\begin{aligned}
\big(\overline{\mathtt{d}\rho}(x)\varphi\big)(s) &= \Big\langle \frac{d}{dt}\Big|_{t=0}\rho(\exp tx)\varphi, K_s\Big\rangle = \frac{d}{dt}\Big|_{t=0}\langle \varphi, \rho((\exp tx)^*)K_s\rangle \\
&= \frac{d}{dt}\Big|_{t=0}\langle \varphi, K_{s\exp tx}\rangle = \frac{d}{dt}\Big|_{t=0}\varphi(s\exp tx) = (\mathcal{L}_x\varphi)(s),
\end{aligned}$$

shows that $\mathcal{L}_x$ extends $\overline{\mathrm{d}\rho}(x)$. Further, for $\psi \in \mathcal{D}_x$,

$$\begin{aligned}
\langle \mathcal{L}_x\psi, K_s\rangle = (\mathcal{L}_x\psi)(s) &= \frac{d}{dt}\Big|_{t=0}\psi(s\exp(tx)) = \Big\langle \psi, \frac{d}{dt}\Big|_{t=0} K_{s\exp tx}\Big\rangle \\
&= \Big\langle \psi, \frac{d}{dt}\Big|_{t=0}\rho(\exp tx)^* K_s\Big\rangle = \langle \psi, \overline{\mathtt{d}\rho}(-\theta(x))K_s\rangle.
\end{aligned}$$

shows that $(\overline{\mathrm{d}\rho}(-\theta(x))|_{\mathcal{H}_K^0})^* = \overline{\mathrm{d}\rho}(x)$ extends $\mathcal{L}_x$. This proves (11). $\square$

The preceding proposition shows that, for any $x \in \mathfrak{h} \cup W$, $\mathcal{L}_x$ leaves the subspace

$$\begin{aligned}
\mathcal{D} &:= \bigcap_{n\in\mathbb{N},\ x_n,\ldots,x_1\in\mathfrak{h}\cup W} \mathcal{D}(\overline{\mathrm{d}\rho}(x_n)\ldots\overline{\mathrm{d}\rho}(x_1)) \\
&= \{\varphi \in \mathcal{H}_K \colon (\forall n \in \mathbb{N})(\forall x_n, \ldots, x_1 \in \mathfrak{h} \cup W)\ \mathcal{L}_{x_1}\cdots\mathcal{L}_{x_n}\varphi \in \mathcal{H}_K\}
\end{aligned}$$

invariant, and by linearity this is also true for $x \in \mathfrak{g}_\mathbb{C}$. Therefore we set

$$\alpha(x) := \mathcal{L}_x|_\mathcal{D} \quad \text{for} \quad x \in \mathfrak{g}_\mathbb{C}$$

and observe that

$$\mathcal{D} = \{\varphi \in \mathcal{H}_K : (\forall n \in \mathbb{N})(\forall x_1, \ldots, x_n \in \mathfrak{g})\ \mathcal{L}_{x_1} \cdots \mathcal{L}_{x_n}\varphi \in \mathcal{H}_K\}. \tag{12}$$

Lemma 4.3. *If $x_1, \ldots x_n \in \mathfrak{g}$, $n \in \mathbb{N}$, then*

$$\mathcal{L}_{x_n} \ldots \mathcal{L}_{x_1}\mathcal{H}_K^0 \subseteq \mathcal{H}_K,$$

and for $s \in S$,

$$\mathcal{L}_{x_n} \ldots \mathcal{L}_{x_1}\rho(s)\varphi_0 = \frac{\partial t^n}{\partial t_n \ldots \partial t_1}\Big|_{t_n=\cdots=t_1=0}\rho(\exp(t_1x_1)\ldots\exp(t_nx_n)s)\varphi_0.$$

Proof. The map

$$(t_1, \ldots, t_n, s) \mapsto \rho(\exp(t_1x_1)\ldots\exp(t_nx_n)s)\varphi_0,$$

defined for small enough t_j, is a smooth $\mathcal{H}_K$-valued map (cf. Remark A.9). Hence

$$\frac{\partial t^n}{\partial t_n \ldots \partial t_1}\Big|_{t_n=\cdots=t_1=0}\rho(\exp(t_nx_n)\ldots\exp(t_1x_1)s)\varphi_0 \in \mathcal{H}_K$$

and

$$\begin{aligned}
&\Big(\frac{\partial t^n}{\partial t_n \ldots \partial t_1}\Big|_{t_n=\cdots=t_1=0}\rho(\exp(t_nx_n)\ldots\exp(t_1x_1)s)\varphi_0\Big)(s_0)\\
&= \frac{\partial t^n}{\partial t_n \ldots \partial t_1}\Big|_{t_n=\cdots=t_1=0}(\rho(s)\varphi_0)(s_0\exp(t_nx_n)\ldots\exp(t_1x_1))\\
&= \big(\mathcal{L}_{x_n} \ldots \mathcal{L}_{x_1}(\rho(s)\varphi_0)\big)(s_0).
\end{aligned}$$

□

Proposition 4.4. *The domain $\mathcal{D}$ contains $\mathcal{H}_K^0$ and is dense in $\mathcal{H}_K$. The map $\alpha : \mathfrak{g}_c \to \mathrm{End}(\mathcal{D})$ is a strongly continuous representation of $\mathfrak{g}_c = \mathfrak{h} \oplus i\mathfrak{q}$ by skew-symmetric operators in the sense that all the maps $\mathfrak{g}_c \to \mathcal{H}_K, x \mapsto \alpha(x)v$ are continuous. More generally, for each $\varphi \in \mathcal{D}^1 := \bigcap_{x\in\mathfrak{g}}\mathcal{D}_x$, the map $\mathfrak{g}_{\mathbb{C}} \to \mathcal{H}_K, x \mapsto \mathcal{L}_x\varphi$ is continuous.*

Proof. The first assertion follows from (12) and Lemma 4.3. The map α is a Lie algebra homomorphism because $x \mapsto \mathcal{L}_x$ is so by (10). For the strong continuity it suffices to show that, for $\varphi \in \mathcal{D}^1$, the graph of the map $\mathfrak{g}_c \to \mathcal{H}_K, x \mapsto \mathcal{L}_x\varphi$ is closed (cf. [Ne10a, Lemma 4.2]). This follows if, for each $s \in S$ the map

$$\mathfrak{g}_c \to \mathcal{H},\ x \mapsto (\mathcal{L}_x\varphi)(s) = d\varphi(s)V_x(s)$$

is continuous. That this is the case on the real Lie algebra $\mathfrak{g}$ follows from the fact that φ is a smooth function on S because $\mathfrak{g} \to T_s(S), x \mapsto V_x(s)$ is a topological isomorphism (cf. Definition A.8). Clearly, the continuity is inherited by the complex linear extension to $\mathfrak{g}_{\mathbb{C}}$. This completes the proof. □

To show that the representation of $\mathfrak{g}_c$ integrates to a continuous unitary representation of G_c we will use the following theorem

Theorem 4.5 ([Mer10]). *Let G_c be a simply connected Banach–Lie group with Lie algebra $\mathfrak{g}_c$. Assume that $\mathfrak{g}_c = \mathfrak{a}_1 \oplus \mathfrak{a}_2$ where $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are closed subspaces. Let α be a strongly continuous representation of $\mathfrak{g}_c$ on a dense subspace $\mathcal{D}$ of a Hilbert space $\mathcal{H}$ such that for every $x \in \mathfrak{a}_1 \cup \mathfrak{a}_2$, $\alpha(x)$ is essentially skew-adjoint, $e^{\overline{\alpha(x)}}\mathcal{D} \subseteq \mathcal{D}$ and*

$$e^{\overline{\alpha(x)}}\alpha(y)e^{-\overline{\alpha(x)}} = \alpha(e^{\mathrm{ad}\,x}y) \quad \text{for} \quad y \in \mathfrak{g}_c.$$

Then α integrates to a continuous unitary representation $(\pi, \mathcal{H})$ of G_c for which $\mathcal{D} \subseteq \mathcal{H}^\infty(\pi)$ is a G_c-invariant subspace and $\alpha(x) = \mathtt{d}\pi(x)|_{\mathcal{D}}$ for $x \in \mathfrak{g}_c$. In particular, for $x \in \mathfrak{g}_c$, the infinitesimal generator $\overline{\mathtt{d}\pi}(x)$ of the one-parameter group $\pi(\exp tx)$ coincides with the closure $\overline{\alpha(x)}$.

The next two propositions ensure that the assumptions of the theorem are satisfied by the representation α for $\mathfrak{a}_1 = \mathfrak{h}$ and $\mathfrak{a}_2 = i\mathfrak{q}$. The first one uses the Geometric Fröhlich Theorem 2.5.

Proposition 4.6. *For $x \in \mathfrak{h} \cup i\mathfrak{q}$, the operator $\alpha(x)$ is essentially skew-adjoint with closure $\mathcal{L}_x^K$.*

Proof. For $x \in \mathfrak{h} \cup iW$, we know that $\mathcal{H}_K^0$ is a core for the skew-adjoint operator $\overline{\mathrm{d}\rho}(x) = \mathcal{L}_x^K$ (Proposition 4.2). Since $\mathcal{H}_K^0 \subseteq \mathcal{D} \subseteq \mathcal{D}_x$, the larger subspace $\mathcal{D}$ is also a core for $\mathcal{L}_x^K$. This proves the assertion for $x \in \mathfrak{h} \cup iW$.

Now let $x \in \mathfrak{q}$ be a general element. Then the vector field V_x is K-symmetric, since for every $s_1, s_2 \in S$, we have

$$K(s_1 \exp tx, s_2) = \varphi_0(s_1(\exp tx)s_2^*) = \varphi_0(s_1(s_2 \exp tx)^*) = K(s_1, s_2 \exp tx).$$

We can therefore apply Theorem 2.5 and Proposition 4.2 to see that $\mathcal{L}_x^K$ is a self-adjoint operator, and that $\mathcal{L}_x|_{\mathcal{H}_K^0}$ is essentially self-adjoint. Writing $x = x_+ - x_-$ with $x_\pm \in W$, we see that we also have $\mathcal{L}_x = \mathcal{L}_{x_+} - \mathcal{L}_{x_-}$, so that

$$\mathcal{H}_K^0 \subseteq \mathcal{D} \subseteq \mathcal{D}_{x_+} \cap \mathcal{D}_{x_-} \subseteq \mathcal{D}_x,$$

and thus $\alpha(ix)$ is essentially skew-adjoint with closure $\mathcal{L}_{ix}^K$. □

Lemma 4.7. *Let $x \in \mathfrak{h} \cup W$, $y \in \mathfrak{g}$, and $\varphi \in \mathcal{H}_K$ such that $\varphi \in \mathcal{D}_{e^{-\operatorname{ad} x}y}$. Then $\rho(\exp x)\varphi \in \mathcal{D}_y$ and*

$$\mathcal{L}_y(\rho(\exp x)\varphi) = \rho(\exp x)\mathcal{L}_{e^{-\operatorname{ad} x}y}\varphi.$$

Proof. For every $s \in S$, $\mathcal{L}_{e^{-\operatorname{ad} x}y}\varphi \in \mathcal{H}$ leads to

$$\begin{aligned}
&\langle \rho(\exp x)\mathcal{L}_{e^{-\operatorname{ad} x}y}\varphi, K_s\rangle \\
&= \langle \mathcal{L}_{e^{-\operatorname{ad} x}y}\varphi, \rho((\exp x)^*)K_s\rangle = \langle \mathcal{L}_{e^{-\operatorname{ad} x}y}\varphi, K_{s\exp x}\rangle \\
&= \frac{d}{dt}\Big|_{t=0}\varphi(s\exp x \exp(te^{-\operatorname{ad} x}y)) = \frac{d}{dt}\Big|_{t=0}\varphi(s\exp ty \exp x) \\
&= \mathcal{L}_y(\rho(\exp x)\varphi)(s) = \langle \mathcal{L}_y(\rho(\exp x)\varphi), K_s\rangle.
\end{aligned}$$ □

Lemma 4.8. *Let $A : \mathcal{D} \to \mathcal{H}$ be a symmetric operator and $\mathcal{D}_1 \subseteq \mathcal{D}$ a dense subspace. If $A_1 := A|_{\mathcal{D}_1}$ is essentially self-adjoint, then A is also essentially self-adjoint, with $\overline{A} = \overline{A_1}$.*

Proof. From $A_1 \subseteq A \subseteq A^*$ it follows that

$$\overline{A} \subseteq A^* \subseteq A_1^* = \overline{A_1} \subseteq \overline{A}$$

(cf. [RS80, Thm. VIII.1]). Therefore $\overline{A} = A^* = \overline{A_1}$ is self-adjoint. □

Proposition 4.9. *For $x \in \mathfrak{h} \cup i\mathfrak{q}$, $e^{\overline{\alpha(x)}}\mathcal{D} \subseteq \mathcal{D}$. Moreover for any $y \in \mathfrak{g}_c$ we have*

$$e^{\overline{\alpha(x)}}\alpha(y)e^{-\overline{\alpha(x)}} = \alpha(e^{\operatorname{ad} x}y). \tag{13}$$

Proof. In the following we also write $\rho(h) := \rho_H(h)$ for $h \in H$.
Step 1: Let $x \in \mathfrak{h} \cup W$ and $\varphi \in \mathcal{D}$. Then Lemma 4.7 implies by induction that for every $n \in \mathbb{N}$ and $y_1, \ldots, y_n \in \mathfrak{h} \cup W$,

$$\mathcal{L}_{y_n} \ldots \mathcal{L}_{y_1}\rho(\exp x)\varphi = \rho(\exp x)\mathcal{L}_{e^{-\operatorname{ad} x}y_n} \ldots \mathcal{L}_{e^{-\operatorname{ad} x}y_1}\varphi \in \mathcal{H},$$

and hence $e^{\overline{\mathrm{d}\rho}(x)}\varphi = \rho(\exp x)\varphi \in \mathcal{D}(\overline{\mathrm{d}\rho}(y_n) \ldots \overline{\mathrm{d}\rho}(y_1))$. It follows that

$$e^{\overline{\mathrm{d}\rho}(x)}\mathcal{D} \subseteq \mathcal{D},$$

and that we have for $\varphi \in \mathcal{D}$ and (by linearity) for $y \in \mathfrak{g}_c$,

$$\alpha(y)e^{\overline{\mathrm{d}\rho}(x)}\varphi = e^{\overline{\mathrm{d}\rho}(x)}\alpha(e^{-\operatorname{ad} x}y)\varphi. \tag{14}$$

In particular (13) holds for $x \in \mathfrak{h}$. Now let $\psi \in \mathcal{D}$. Then (14) can be written

$$\langle -e^{\overline{\mathrm{d}\rho}(x)}\varphi, \alpha(y)\psi\rangle = \langle e^{\overline{\mathrm{d}\rho}(x)}\alpha(e^{-\operatorname{ad} x}y)\varphi, \psi\rangle, \tag{15}$$

and this last equation is all we need for the following.

Step 2: Assume that $x \in W$. Then the spectrum of $\overline{\mathrm{d}\rho}(x)$ is bounded from above and hence $t \mapsto e^{t\overline{\mathrm{d}\rho}(x)}$ extends to a strongly continuous holomorphic semigroup

$$\mathbb{C}^+ = \{z \in \mathbb{C} \mid \operatorname{Re} z \geq 0\} \to B(\mathcal{H}),\ z \mapsto e^{z\overline{\mathrm{d}\rho}(x)},$$

which is holomorphic on $\operatorname{int}(\mathbb{C}^+)$ (cf. [HN93, Prop. 9.9] or [Ne00, Prop. VI.3.2]). For $\varphi \in \mathcal{D}^1$, the map $\mathfrak{g}_{\mathbb{C}} \to \mathcal{H}$, $x \mapsto \mathcal{L}_x\varphi$ is $\mathbb{C}$-linear and continuous (Proposition 4.4), hence the function $\mathbb{C} \to \mathcal{H}$, $z \mapsto \mathcal{L}_{e^{-z\operatorname{ad} x}y}\varphi$ is analytic. Since the map

$$B(\mathcal{H}) \times \mathcal{H} \to \mathcal{H}, \quad (T, \xi) \mapsto T\xi$$

is $\mathbb{C}$-bilinear and continuous, it follows that the map

$$\operatorname{int}(\mathbb{C}^+) \to \mathcal{H}, \quad z \mapsto e^{z\overline{\mathrm{d}\rho}(x)}\mathcal{L}_{e^{-z\operatorname{ad} x}y}\varphi$$

is analytic. By the Analytic Continuation Principle, the equality (15) implies

$$\langle -e^{z\overline{\mathrm{d}\rho}(x)}\varphi, \alpha(y)\psi\rangle = \langle e^{z\overline{\mathrm{d}\rho}(x)}\mathcal{L}_{e^{-z\operatorname{ad} x}y}\varphi, \psi\rangle \quad \text{for } z \in \operatorname{int}\mathbb{C}^+, \psi \in \mathcal{D}.$$

We then have by continuity

$$\langle -e^{\pm i\overline{\mathrm{d}\rho}(x)}\varphi, \alpha(y)\psi\rangle = \langle e^{\pm i\overline{\mathrm{d}\rho}(x)}\mathcal{L}_{e^{\mp i\operatorname{ad} x}y}\varphi, \psi\rangle. \tag{16}$$

This shows that, for $y \in \mathfrak{h} \cup iW$, $e^{\pm i\overline{\mathrm{d}\rho}(x)}\varphi \in \mathcal{D}(\alpha(y)^*) = \mathcal{D}(\overline{\mathrm{d}\rho}(y)) = \mathcal{D}_y$. We thus arrive at

$$e^{\pm i\overline{\mathrm{d}\rho}(x)}\mathcal{D}^1 \subseteq \mathcal{D}^1 \quad \text{with} \quad \mathcal{L}_y e^{\pm i\overline{\mathrm{d}\rho}(x)} = e^{\pm i\overline{\mathrm{d}\rho}(x)}\mathcal{L}_{e^{\mp i\operatorname{ad} x}y} \quad \text{on} \quad \mathcal{D}^1.$$

By induction, we now obtain

$$e^{\pm i\overline{\mathrm{d}\rho}(x)}\mathcal{D} = \mathcal{D} \quad \text{with} \quad \alpha(y)e^{\pm i\overline{\mathrm{d}\rho}(x)}|_{\mathcal{D}} = e^{\pm i\overline{\mathrm{d}\rho}(x)}\alpha(e^{\mp i\operatorname{ad} x}y).$$

Step 3: For $n \in \mathbb{N}$, $x_1, x_2 \in W$, $y \in \mathfrak{g}_c$ and $\varphi, \psi \in \mathcal{D}^1$ we now obtain

$$\begin{aligned}\langle -\left(e^{i\overline{\mathrm{d}\rho}(x_1)}e^{-i\overline{\mathrm{d}\rho}(x_2)}\right)^n\varphi, \overline{\mathrm{d}\rho}(y)\psi\rangle \\ = \langle \mathcal{L}_{\left(e^{i\operatorname{ad} x_2}e^{-i\operatorname{ad} x_1}\right)^n y}\varphi, \left(e^{i\overline{\mathrm{d}\rho}(x_2)}e^{-i\overline{\mathrm{d}\rho}(x_1)}\right)^n\psi\rangle\end{aligned} \tag{17}$$

from

$$\mathcal{L}_y\left(e^{i\overline{\mathrm{d}\rho}(x_1)}e^{-i\overline{\mathrm{d}\rho}(x_2)}\right)^n\varphi = \left(e^{i\overline{\mathrm{d}\rho}(x_1)}e^{-i\overline{\mathrm{d}\rho}(x_2)}\right)^n\mathcal{L}_{\left(e^{i\operatorname{ad} x_2}e^{-i\operatorname{ad} x_1}\right)^n y}\varphi.$$

Now let $x \in \mathfrak{q}$ and write it as $x = x_1 - x_2$ with $x_1, x_2 \in W$. Since $\overline{\mathrm{d}\rho}(x_1) - \overline{\mathrm{d}\rho}(x_2)$ is essentially self-adjoint on $\mathcal{D}$ (Proposition 4.6), it is essentially self-adjoint as an operator on its domain $\mathcal{D}(\overline{\mathrm{d}\rho}(x_1)) \cap \mathcal{D}(\overline{\mathrm{d}\rho}(-x_2))$, and its closure is $\overline{\alpha(x_1 - x_2)}$ (Lemma 4.8). We therefore have Trotter's Product Formula [RS80, Thm. VIII.31]:

$$\lim_{n\to\infty}\left(e^{i\frac{\overline{\mathrm{d}\rho}(x_1)}{n}}e^{-i\frac{\overline{\mathrm{d}\rho}(x_2)}{n}}\right)^n f = e^{i\overline{\alpha(x_1 - x_2)}}f \quad \text{for} \quad f \in \mathcal{H}_K. \tag{18}$$

Replacing x_j by $\frac{x_j}{n}$, $j = 1, 2$, in (17) and taking the limit we obtain

$$\langle -e^{i\overline{\alpha(x_1 - x_2)}}\varphi, \mathcal{L}_y\psi\rangle = \langle e^{i\overline{\alpha(x_1 - x_2)}}\mathcal{L}_{e^{i\operatorname{ad}(x_2 - x_1)}y}\varphi, \psi\rangle \quad \text{for} \quad \varphi, \psi \in \mathcal{D}^1. \tag{19}$$

As above we now obtain by induction that $e^{i\overline{\alpha(x)}}\mathcal{D} \subseteq \mathcal{D}$ and that (13) holds for $x \in i\mathfrak{q}$. $\square$

Proof of Theorem 3.7: In view of Proposition 4.9, we obtain from Theorem 4.5 a unitary representation of G_c whose space of smooth vectors contains $\mathcal{D}$ and such that $\mathrm{d}\pi|_{\mathcal{D}} = \alpha$ and $\overline{\mathrm{d}\pi(x)} = \overline{\alpha(x)}$ for $x \in \mathfrak{g}_c$. We conclude with Proposition 4.2 and 4.6 that $\overline{\mathrm{d}\pi}(x) = \overline{\mathrm{d}\rho}(x)$ for $x \in \mathfrak{h}$ and $\overline{\mathrm{d}\pi}(ix) = i\overline{\mathrm{d}\rho}(x)$ for $x \in W$. □

Remark 4.10. Since $\mathcal{D} \subseteq \mathcal{H}^\infty(\pi)$, it immediatly follows from

$$\mathcal{H}^\infty(\pi) = \bigcap_{x_j \in \mathfrak{g}_c,\ n \in \mathbb{N}} \mathcal{D}(\overline{\mathrm{d}\pi}(x_n) \ldots \overline{\mathrm{d}\pi}(x_1)) \subseteq \mathcal{D}$$

([Ne10b, Lemma 3.4, Remark 8.3]), that $\mathcal{D} = \mathcal{H}^\infty(\pi)$.

5. Holomorphic extension of semibounded representations

In this section we obtain a result which is a converse to the Lüscher–Mack Theorem. It is new even in the finite dimensional setting. At the same time, we obtain the existence of holomorphic extensions for semibounded unitary representations.

Definition 5.1. Let G be a Banach–Lie group with Lie algebra $\mathfrak{g}$. For a smooth unitary representation $(\pi, \mathcal{H})$ of G we consider the map

$$s_\pi : \mathfrak{g} \to \mathbb{R} \cup \{\infty\}, \quad s_\pi(x) := \sup\big(\mathrm{Spec}(i\overline{\mathrm{d}\pi}(x))\big).$$

(a) A smooth unitary representation $(\pi, \mathcal{H})$ of G is called *semibounded* if s_π is bounded on a non-empty open subset of $\mathfrak{g}$. Then the cone W_π, consisting of all those $x \in \mathfrak{g}$ for which s_π is bounded on a neighborhood of x, is an open $\mathrm{Ad}(G)$-invariant convex cone in $\mathfrak{g}$. Moreover, $s_\pi \colon W_\pi \to \mathbb{R}$ is a continuous convex function ([Ne08]).

For a convex cone $W \subseteq \mathfrak{g}$, we say that π is W*-semibounded* if $s_\pi(W) \subseteq \mathbb{R}$ and $s_\pi \colon W \to \mathbb{R}$ is locally bounded.

(b) A convex cone $W \subseteq \mathfrak{g}$ is said to be *relatively open* if the linear subspace $W - W$ of $\mathfrak{g}$ is closed and W is open in $W - W$.

For the definition of a C^1-map we use in the next lemma see Definition B.2.

Lemma 5.2. *Let W be a relatively open convex cone in $\mathfrak{g}$, $\mathfrak{q} := W - W$, and $(\pi, \mathcal{H})$ be a smooth W-semibounded unitary representation of G. Then, for every $v \in \mathcal{H}^\infty(\pi)$, the map*

$$\rho^v \colon W \to \mathcal{H},\ x \mapsto e^{i\overline{\mathrm{d}\pi}(x)}v,$$

is C^1 and

$$T_x(\rho^v)(y) = \overline{\mathrm{d}\pi}\left(\int_0^1 e^{s\,\mathrm{ad}\,ix} y\,ds\right) e^{i\overline{\mathrm{d}\pi}(x)}v.$$

The map $G \times W \to \mathcal{H}, (g, x) \mapsto \pi(g)\rho^v(x)$ is also C^1.

Proof. For $x \in W$, $y \in \mathfrak{g}_c$ and $v, w \in \mathcal{H}^\infty$, the function

$$F(z) := \langle -e^{z\overline{\mathrm{d}\pi}(x)}v, \mathrm{d}\pi(y)w\rangle - \langle e^{z\overline{\mathrm{d}\pi}(x)}\mathrm{d}\pi(e^{-z\,\mathrm{ad}\,x}y)v, w\rangle$$

is continuous on the closed upper half-plane $\mathbb{C}_+ := \{z \in \mathbb{C} \colon \operatorname{Im} z \geq 0\}$ and holomorphic on its interior ([Ne00, Prop. VI.3.2]; see also Step 2 in the proof of Proposition 4.9). Since $F(t) = 0$ for $t \in \mathbb{R}$ by Proposition 4.9, the Schwarz Reflection Principle [Ru87, Thm. 11.14] implies that F vanishes on $\mathbb{C}_+$. It follows that, for $x \in W$ and $v \in \mathcal{H}^\infty$ we have $e^{i\overline{\mathrm{d}\pi}(x)}v \in \mathcal{D}\big(\overline{\mathrm{d}\pi(y)}\big)$ for every $y \in \mathfrak{g}$ with

$$\overline{\mathrm{d}\pi}(y)e^{i\overline{\mathrm{d}\pi}(x)}v = e^{i\overline{\mathrm{d}\pi}(x)}\mathrm{d}\pi(e^{-\mathrm{ad}\,ix}y)v.$$

Since $\mathrm{d}\pi(e^{-\mathrm{ad}\,ix}y)v$ is again a smooth vector, we can iterate this argument to obtain inductively

$$e^{i\overline{\mathrm{d}\pi}(x)}v \in \mathcal{D} := \bigcap_{y_n, \ldots, y_1 \in \mathfrak{g},\ n \in \mathbb{N}} \mathcal{D}(\overline{\mathrm{d}\pi}(y_n)) \ldots \mathcal{D}(\overline{\mathrm{d}\pi}(y_1)).$$

We know by [Ne10b, Lemma 3.4, Remark 8.3] that $\mathcal{D} = \mathcal{H}^\infty$. Hence we have

$$e^{i\overline{\mathrm{d}\pi}(x)}\mathcal{H}^\infty \subseteq \mathcal{H}^\infty \text{ and } \mathrm{d}\pi(y)e^{i\overline{\mathrm{d}\pi}(x)} = e^{i\overline{\mathrm{d}\pi}(x)}\mathrm{d}\pi(e^{-\operatorname{ad} ix}y) \tag{20}$$

for $x \in W$ and $y \in \mathfrak{g}_\mathbb{C}$. Now let $v \in \mathcal{H}^\infty$. In view of (20), Proposition B.5 implies that ρ^v is C^1 with

$$T_x(\rho^v)(y) = \mathrm{d}\pi\left(\int_0^1 e^{s\operatorname{ad} ix}y ds\right)e^{i\overline{\mathrm{d}\pi}(x)}v = e^{i\overline{\mathrm{d}\pi}(x)}\mathrm{d}\pi\left(\int_0^1 e^{-s\operatorname{ad} ix}y ds\right)v$$

and that the map

$$\widehat{\rho}\colon W \times \mathcal{H} \to \mathcal{H}, \quad (x, v) \mapsto e^{i\overline{\mathrm{d}\pi}(x)}v$$

is continuous.

Finally, we observe that the map

$$F\colon G \times W \to \mathcal{H}, \quad F(g, x) := \pi(g)\rho^v(x)$$

is continuous because the G-action on $\mathcal{H}$ defined by π is continuous. We have just seen that F is partially differentiable in x with continuous partial derivative

$$G \times W \times \mathfrak{q} \to \mathcal{H}, \quad (g, x, y) \mapsto \pi(g)T_x(\rho^v)y.$$

We have also seen that $\rho^v(W) \subseteq \mathcal{H}^\infty(\pi)$, so that the partial derivatives in g also exist and are given by

$$TG \times W \to \mathcal{H}, \quad (g.y, x) \mapsto \pi(g)\mathtt{d}\pi(y)e^{i\overline{\mathtt{d}\pi}(x)}v = \pi(g)e^{i\overline{\mathtt{d}\pi}(x)}\mathtt{d}\pi(e^{-\operatorname{ad}(ix)}y)v.$$

As G acts continously on $\mathcal{H}$, $\widehat{\rho}$ is continuous, and the adjoint action of G on $\mathfrak{g}$ is continuous, this function is continuous. This implies that F is C^1 (cf. [Ham82]). $\square$

Definition 5.3. Let W be a relatively open convex cone in $\mathfrak{g}$, $\mathfrak{q} := W - W$, and $\mathfrak{h}$ be a closed subalgebra of $\mathfrak{g}$. The cone W is called $\mathfrak{h}$-*compatible* if

$$[W, W] \subseteq \mathfrak{h} \quad \text{and} \quad e^{\operatorname{ad}\mathfrak{h}}W \subseteq W.$$

Then $\mathfrak{g}_c := \mathfrak{h} \oplus i\mathfrak{q} \subseteq \mathfrak{g}_\mathbb{C}$ is a closed subalgebra which is turned in a symmetric Banach–Lie algebra by the involution $\theta(x + iy) := x - iy$ for $x \in \mathfrak{h}, y \in \mathfrak{q}$.

An $\mathfrak{h}$-compatible cone $W \subseteq \mathfrak{g}$ is said to be *integrable* if there exists a symmetric Banach–Lie group (G_c, θ) with symmetric Lie algebra $\mathfrak{g}_c = \mathfrak{h} \oplus i\mathfrak{q}$ such that, for $H := (G_c^\theta)_0$, the polar map

$$H \times iW \to G_c, \quad (h, x) \mapsto h\exp(x)$$

is an analytic diffeomorphism onto an open subsemigroup

$$S = S_H(iW) = H\exp(iW) \subseteq G.$$

Then S is invariant under the involution $s^* = \theta(s)^{-1}$, turning it into an involutive semigroup $(S, *)$.

From the discussion of Banach–Olshanski semigroups in Appendix A, it then follows that for each connected Banach–Lie group H_1 locally isomorphic to H to which the adjoint action of $\mathfrak{h}$ on $\mathfrak{g}_c$ integrates, there exists an involutive Banach–Olshanski semigroup $S_{H_1}(iW)$ with a polar decomposition which is a quotient of the universal covering semigroup of $S_H(iW)$.

Theorem 5.4. *Let G be a Banach–Lie group with Lie algebra $\mathfrak{g}$, $\mathfrak{h}$ be a closed complemented Lie subalgebra of $\mathfrak{g}$, and $H := \langle\exp\mathfrak{h}\rangle$ be the corresponding integral subgroup in G. Let $(\pi, \mathcal{H})$ be a smooth W-semibounded unitary representation of G for the integrable $\mathfrak{h}$-compatible cone W. Then the formula*

$$\rho(h\exp ix) := \pi(h)e^{i\overline{\mathrm{d}\pi}(x)} \textit{ for } h \in H \textit{ and } x \in W,$$

defines a strongly continuous smooth $*$-representation ρ of $S_H(iW)$ on $\mathcal{H}$.

We shall need the following Chain Rule ([Mer10, Lemma 4]):

Lemma 5.5. *Let $I \subseteq \mathbb{R}$ be an open interval, E and F be two Banach spaces and $L_s(E,F)$ denotes the space of continuous linear operators from E to F endowed with the strong operator topology. Let $I \to L_s(E,F), t \mapsto K(t)$ be a continuous path such that $t \mapsto K(t)v$ is differentiable for every v in a subspace $\mathcal{D}$ of E and let $\gamma(t)$ be a differentiable path in $\mathcal{D}$. We write $K'(t) : \mathcal{D} \to F$ for the linear operator obtained by $K'(t)v := \frac{d}{dt}K(t)v$ for $v \in \mathcal{D}$. Then $t \mapsto K(t)\gamma(t)$ is differentiable with*

$$\frac{d}{dt}K(t)\gamma(t) = K'(t)\gamma(t) + K(t)\gamma'(t).$$

Proof of Theorem 5.4. **Step 1:** Let us first prove that, for $x_1, x_2 \in W$,

$$\rho(\exp ix_1 \exp ix_2) = \rho(\exp ix_1)\rho(\exp ix_2). \tag{21}$$

For this purpose, let us write for $t > 0$,

$$\eta(t) := \exp tix_1 \exp ix_2 = h_t \exp(ix(t)) \text{ with } h_t \in H,\ x(t) \in W. \tag{22}$$

Now let $v \in \mathcal{H}^\infty$ and consider $\gamma(t) := \rho(\eta(t))v = \pi(h_t)e^{i\overline{\mathtt{d}\pi}(x(t))}v$. By Lemma 5.2 and the Chain Rule (Lemma 5.5), applied with $K(t) = \pi(h_t)$ and $\mathcal{D} = \mathcal{H}^\infty(\pi)$, the path $\gamma(t)$ is differentiable for $t > 0$, and, denoting by δ the right logarithmic derivative (see Definition A.11), we obtain with Proposition A.12

$$\begin{aligned}
\gamma'(t) &= \overline{\mathtt{d}\pi}(\delta h_t)\gamma(t) + \pi(h_t)\overline{\mathtt{d}\pi}\Big(\int_0^1 e^{is\,\mathrm{ad}\,x(t)}ix'(t)ds\Big)e^{\overline{\mathtt{d}\pi}(ix(t))}v \\
&= \overline{\mathtt{d}\pi}(\delta h_t)\gamma(t) + \pi(h_t)\overline{\mathtt{d}\pi}\big(\delta(\exp)_{ix(t)}ix'(t)\big)e^{\overline{\mathtt{d}\pi}(ix(t))}v \\
&= \overline{\mathtt{d}\pi}\Big(\delta(h)_t + \mathrm{Ad}(h_t)\delta(\exp)_{ix(t)}ix'(t)\Big)\gamma(t) \\
&= \overline{\mathtt{d}\pi}(\delta(\eta)_t)\gamma(t) \qquad \text{(by Proposition A.13)} \\
&= \overline{\mathtt{d}\pi}(ix_1)\gamma(t).
\end{aligned}$$

Since $\lim_{t\to 0}\gamma(t) = \lim_{t\to 0}\pi(h_t)e^{i\overline{\mathtt{d}\pi}(x(t))}v = \rho(\exp ix_2)v$ (Lemma 5.2, Lemma A.6), we obtain with [Kat66, p. 481] that

$$\gamma(t) = \rho(\exp tix_1)\rho(\exp ix_2)v,$$

and (21) follows for $t = 1$.

Step 2: For $h \in H$ and $x \in W$ we have

$$\pi(h)\overline{\mathtt{d}\pi}(x)\pi(h)^{-1} = \overline{\mathtt{d}\pi}(\mathrm{Ad}(h)x),$$

so that

$$e^{\overline{\mathtt{d}\pi}(i\,\mathrm{Ad}(h)x)} = e^{\pi(h)i\overline{\mathtt{d}\pi}(x)\pi(h)^{-1}} = \pi(h)e^{i\overline{\mathtt{d}\pi}(x)}\pi(h)^{-1}. \tag{23}$$

From (23) we obtain the relation

$$\rho(h\exp x)^* = \rho((h\exp x)^*),$$

and we further derive

$$\rho(sh) = \rho(s)\pi(h) \quad \text{for} \quad s \in S, h \in H. \tag{24}$$

Step 3: Now we can prove that for $h_1, h_2 \in H$ and $x_1, x_2 \in W$,

$$\rho(h_1 \exp ix_1 h_2 \exp ix_2) = \rho(h_1 \exp ix_1)\rho(h_2 \exp ix_2). \tag{25}$$

With (21) and (23) we obtain

$$\begin{aligned}\rho(h_1 \exp ix_1)\rho(h_2 \exp ix_2) &= \pi(h_1)e^{\overline{\mathrm{d}\pi}(ix_1)}\pi(h_2)e^{i\overline{\mathrm{d}\pi}(x_2)} \\ &= \pi(h_1)\pi(h_2)e^{i\overline{\mathrm{d}\pi}(\mathrm{Ad}(h_2)^{-1}x_1)}e^{i\overline{\mathrm{d}\pi}(x_2)} \\ &= \pi(h_1h_2)\rho(\exp(i\,\mathrm{Ad}(h_2)^{-1}x_1))\rho(\exp(ix_2)) \\ &= \pi(h_1h_2)\rho(\exp(i\,\mathrm{Ad}(h_2)^{-1}x_1)\exp(ix_2)) \\ &= \rho(h_1h_2\exp(i\,\mathrm{Ad}(h_2)^{-1}x_1)\exp(ix_2)) \\ &= \rho(h_1 \exp ix_1)h_2 \exp ix_2).\end{aligned}$$

Step 4: It remains to prove that for $v \in \mathcal{H}^\infty(\pi)$ the map $\rho^v : S \to \mathcal{H}$, $\rho^v(s) := \rho(s)v$ is smooth. In view of Lemma 5.2, this map is C^1. For $x \in \mathfrak{h}$ we have by Remark A.9 and (24)

$$T_s(\rho^v)(s.x) := \frac{d}{dt}\Big|_{t=0}\rho(s\exp tx)v = \frac{d}{dt}\Big|_{t=0}\rho(s)\rho(\exp tx)v = \rho(s)\mathrm{d}\pi(x)v.$$

Similarly we have for $x \in W$

$$T_s(\rho^v)(s.(ix)) := \frac{d}{dt}\Big|_{t=0}\rho(s\exp tix)v = \frac{d}{dt}\Big|_{t=0}\rho(s)\rho(\exp tix)v = \rho(s)\mathrm{d}\pi(ix)v.$$

Since $T_s(\rho^v)$ linear, it follows that

$$T_s(\rho^v)(s.x) = \rho(s)\mathrm{d}\pi(x)v \quad \text{for } x \in \mathfrak{h} + i\mathfrak{q}. \tag{26}$$

Now an easy induction shows that the higher partial derivatives of $T\rho^v$ only involve the continuous n-linear maps

$$\omega_v^n(x_1, \ldots, x_n) := \mathrm{d}\pi(x_1)\ldots\mathrm{d}\pi(x_n)v,$$

and hence that ρ^v is smooth. □

Now recall the context of the Lüscher–Mack Theorem. We have a symmetric Banach–Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ and an integrable $e^{\mathrm{ad}\,\mathfrak{h}}$-invariant open convex cone $W \subseteq \mathfrak{q}$. We therefore have a Banach–Olshanski semigroup $S_H(W) = H \exp W$ for each connected Lie group with Lie algebra $\mathfrak{h}$. Applying the preceding theorem to $\mathfrak{g}_c = \mathfrak{h} + i\mathfrak{q}$ and $-iW \subseteq i\mathfrak{q}$, we obtain the following converse to the Lüscher–Mack Theorem:

Corollary 5.6. *Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ be a symmetric Banach–Lie algebra and W be an integrable $e^{\mathrm{ad}\,\mathfrak{h}}$-invariant open convex cone in $\mathfrak{q}$. Let G_c be a Banach–Lie group with Lie algebra $\mathfrak{g}_c = \mathfrak{h} + i\mathfrak{q} \subseteq \mathfrak{g}_{\mathbb{C}}$ and let H_c be its integral subgroup with Lie algebra $\mathfrak{h}$. Let π be an $-iW$-semibounded unitary representation of G_c. Then*

$$\rho(h\exp x) := \pi(h)e^{\overline{\mathrm{d}\pi}(x)} \text{ for } h \in H_c \text{ and } x \in W,$$

defines a strongly continuous smooth $*$-representation ρ of $S_{H_c}(W)$.

Theorem 5.7 (Holomorphic Extension Theorem)**.** *Let G be a Banach-Lie group with Lie algebra $\mathfrak{g}$, $(\pi, \mathcal{H})$ be a semibounded unitary representation of G, and $W \subseteq W_\pi$ be an open integrable* $\mathrm{Ad}(G)$*-invariant convex cone. Then*

$$\rho(g\exp ix) := \pi(g)e^{i\overline{\mathrm{d}\pi}(x)} \text{ for } g \in G \text{ and } x \in W,$$

defines a holomorphic $*$-representation ρ of the complex involutive semigroup

$$S_G(iW) = G\exp iW.$$

In particular the vectors in $\rho(S_G(iW))\mathcal{H}$ are analytic for π.

Note that $S_G(iW)$ is a complex Olshanski semigroup (cf. Definition A.4).

Proof. First we observe that the cone W is $\mathfrak{g}$-compatible. Theorem 5.4 now applies to the semigroup $S_G(iW)$. It remains to prove that $\rho : S_G(iW) \to B(\mathcal{H})$ is holomorphic. But (26) shows that $T\rho^v$ is complex linear, hence that ρ^v is holomorphic. The holomorphy of ρ now follows from [Ne00, Lemma IV.2.2]. Now let $s \in S_G(iW)$ and $v \in \mathcal{H}$. Since $\pi^{\rho(s)v}(h) = \pi(h)\rho(s)v = \rho(hs)v$, the analyticity of $\pi^{\rho(s)v}$ follows from the analyticity of the map $H \to S$, $h \mapsto hs$. □

The preceding theorem generalizes Olshanski's Holomorphic Extension Theorem for highest weight representations ([Ol82], [Ne00]) to the Banach–Lie setting. In the finite dimensional case the proof heavily relies on the existence of a dense space of analytic vectors, which can be derived by convolution with heat kernels ([Ga60]), but for unitary representations of Banach–Lie groups, not even the space of C^1-vectors need to be dense (cf. [Ne10b]). In the finite dimensional context one proves first that ρ is holomorphic, and then the multiplicativity of ρ is obtained by analytic continuation. In the proof we give here the multiplicativity of ρ follows from the (assumed) existence of smooth vectors and then the holomorphy of ρ follows as a bonus from its multiplicativity.

Appendix A. Covering theory for Olshanski semigroups

Let (G, θ) be a symmetric Banach–Lie group and $(\mathfrak{g}, \theta)$ the corresponding symmetric Lie algebra. We write

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q} \quad \text{with} \quad \mathfrak{h} = \ker(\theta - \mathbf{1}) \quad \text{and} \quad \mathfrak{q} = \ker(\theta + \mathbf{1}),$$

for the eigenspace decomposition of $\mathfrak{g}$ under θ and let H_G denote the identity component of G^θ. We set $g^* = \theta(g)^{-1}$ and consider an open convex $\mathrm{Ad}(H_G)$-invariant convex cone $W \subseteq \mathfrak{q}$ for which the map

$$H_G \times W \to G, \quad (h, x) \mapsto h \exp x$$

is an analytic diffeomorphism onto an open subsemigroup $S = H \exp(W)$ of G. In these coordinates the involution on $h \exp x \in S$ is given by

$$(h \exp x)^* = (\exp x)h^{-1} = h^{-1} \exp(\mathrm{Ad}(h)x). \tag{27}$$

In particular S is $*$-invariant. In the following we write $S_{H_G}(W) = H_G \exp(W)$ for this involutive semigroup.

Definition A.1. Let S be an involutive semigroup. A *multiplier* of S is a pair (λ, ρ) of maps $\lambda, \rho \colon S \to S$ satisfying

$$a\lambda(b) = \rho(a)b, \qquad \lambda(ab) = \lambda(a)b, \qquad \text{and} \qquad \rho(ab) = a\rho(b) \quad \text{for} \quad a, b \in S.$$

We write $M(S)$ for the set of all multipliers of S and turn it into an involutive semigroup by

$$(\lambda, \rho)(\lambda', \rho') := (\lambda \circ \lambda', \rho' \circ \rho) \quad \text{and} \quad (\lambda, \rho)^* := (\rho^*, \lambda^*),$$

where $\lambda^*(a) := \lambda(a^*)^*$ and $\rho^*(a) = \rho(a^*)^*$. We write

$$\mathrm{U}(M(S)) := \{(\lambda, \rho) \in M(S) \colon (\lambda, \rho)(\lambda, \rho)^* = (\lambda, \rho)^*(\lambda, \rho) = \mathbf{1}\}$$

for the *unitary group* of $M(S)$.

Note that $S \to M(S), s \mapsto (\lambda_s, \rho_s)$ is a morphism of involutive semigroups and that $M(S)$ acts on S from the left by $(\lambda, \rho).s := \lambda(s)$ and from the right by $s.(\lambda, \rho) := \rho(s)$.

The group H_G is in general not contained in $S_{H_G}(W)$, but it acts on it by the unitary multipliers (λ_h, ρ_h), $h \in H_G$.

Proposition A.2. *Let $q : \widetilde{S} \to S$ be the universal covering of the Banach manifold $S = S_{H_G}(W)$. Then $\widetilde{S}$ carries the structure of an analytic Banach $*$-semigroup such that the covering map $q\colon \widetilde{S} \to S$ is a homomorphism of Banach $*$-semigroups.*

Moreover, the simply connected covering group $\widetilde{H}_G$ of H_G acts on $\widetilde{S}$ by unitary multipliers and we thus obtain an analytic diffeomorphism

$$\widetilde{\Phi}\colon \widetilde{H}_G \times W \to \widetilde{S}, \quad (h, x) \mapsto h \exp x,$$

where $\exp\colon W \to \widetilde{S}$ *is a continuous lift of* $\exp\colon W \to S$ *such that*

$$\exp(x)^* = \exp(x) \quad \textit{for} \quad x \in W,$$

and

$$\exp(sx)\exp(tx) = \exp((t+s)x) \quad \textit{for} \quad x \in W, t, s > 0. \tag{28}$$

Proof. Since the polar map $\Phi\colon H_G \times W \to S$ is an analytic diffeomorphism, there exists an analytic diffeomorphism $\widetilde{\Phi}\colon \widetilde{H}_G \times W \to \widetilde{S}$ with $q \circ \widetilde{\Phi} = \Phi \circ q$. We then define $\widetilde{\exp}\colon W \to \widetilde{S}, x \mapsto \widetilde{\Phi}(e, x)$.

Pick $x_0 \in W$ and let $\widetilde{m}\colon \widetilde{S} \times \widetilde{S} \to \widetilde{S}, (s,t) \mapsto st$ be the unique continuous lift of the multiplication map $m\colon S \times S \to S$ with

$$\widetilde{\exp}(x_0)\widetilde{\exp}(x_0) = \widetilde{\exp}(2x_0).$$

Then the uniqueness of lifts implies that the restriction of $\widetilde{m}$ to $\widetilde{\exp}(\mathbb{R}_{>0}x_0)$ satisfies

$$\widetilde{\exp}(tx_0)\widetilde{\exp}(sx_0) = \widetilde{\exp}((t+s)x_0)$$

and we obtain in particular

$$(\widetilde{\exp}(x_0)\widetilde{\exp}(x_0))\widetilde{\exp}(x_0) = \widetilde{\exp}(x_0)(\widetilde{\exp}(x_0)\widetilde{\exp}(x_0)).$$

Therefore the uniqueness of lifts implies that $\widetilde{m}$ is associative, hence defines on $\widetilde{S}$ an analytic semigroup structure.

We also lift the involution on S to the unique involutive diffeomorphism $*$ on $\widetilde{S}$ with $\widetilde{\exp}(x_0)^* = \widetilde{\exp}(x_0)$, and since $(st)^* = t^*s^*$ now holds for $s = t = \widetilde{\exp}(x_0)$, the uniqueness of lifts implies that $(\widetilde{S}, *)$ is an involutive semigroup.

The multiplication on S can be expressed by analytic maps $m_{H_G}: W \times W \to H_G$ and $m_W: W \times W \to W$ as

$$\Phi(h,x)\Phi(h',x') = \Phi(hh'm_{H_G}(\mathrm{Ad}(h')^{-1}x, x'), m_W(\mathrm{Ad}(h')^{-1}x, x')). \tag{29}$$

From the continuity of the multiplication $S \times (S \cup H_G) \to S$ in G, it follows that both maps m_W and m_H extends continuously to the set

$$W_2 := \big(W \times (W \cup \{0\})\big) \cup \big((W \cup \{0\}) \times W\big).$$

If $\widetilde{m}_{H_G}: W_2 \to \widetilde{H}_G$ is the unique lift of m_{H_G} satisfying $\widetilde{m}_{H_G}(x_0, x_0) = e$, then we obtain the formula

$$\widetilde{\Phi}(h,x)\widetilde{\Phi}(h',x') = \widetilde{\Phi}(hh'\widetilde{m}_{H_G}(\mathrm{Ad}(h')^{-1}x, x'), m_W(\mathrm{Ad}(h')^{-1}x, x')). \tag{30}$$

For each $x \in W$ and $t, s > 0$ we further have $\widetilde{m}_{H_G}(sx, tx) = e$ because $m_{H_G}(sx, tx) = e$ and the subset $\{(sx, tx) \in W \times W : s, t > 0, x \in W\}$ is connected. This implies (28).

The left and right multiplier actions of H_G on S lift to unique left and right actions of $\widetilde{H}_G$ on $\widetilde{S}$, satisfying

$$h\widetilde{\Phi}(h', x) = \widetilde{\Phi}(hh', x)' \quad \text{and} \quad \widetilde{\Phi}(h', x)h = \widetilde{\Phi}(h'h, \mathrm{Ad}(h)^{-1}x). \tag{31}$$

From (31) we further derive

$$(h\widetilde{\exp}x)(h'\widetilde{\exp}x') = hh'\widetilde{\exp}(\mathrm{Ad}(h')^{-1}x)\widetilde{\exp}(x'). \tag{32}$$

This implies in particular that the left action of $\widetilde{H}_G$ on $\widetilde{S}$ commutes with the right multiplications. We also obtain from the uniqueness of lifts that

$$(h\widetilde{\exp}x)^* = h^{-1}\widetilde{\exp}(\mathrm{Ad}(h)x) = \widetilde{\exp}(x)h^{-1} \quad \text{for} \quad x \in W, h \in \widetilde{H}_G,$$

so that left multiplications in $\widetilde{S}$ commute with the right action of $\widetilde{H}_G$. To see that $\widetilde{H}_G$ acts on $\widetilde{S}$ by unitary multipliers, it remains to observe that

$$(h'\widetilde{\exp}(x')h)(h''\widetilde{\exp}(x'')) = (h'\widetilde{\exp}(x'))(hh''\widetilde{\exp}(x''))$$

for $h, h', h'' \in \widetilde{H}_G, x, x' \in W$, which also follows from (32). $\square$

Let $\mathrm{Ad}_{\mathfrak{q}}^{\widetilde{H}_G} := \mathrm{Ad}_{\mathfrak{q}} \circ q_{H_G}$ be the action of $\widetilde{H}_G$ on $\mathfrak{q}$, obtained from the action $\mathrm{Ad}_{\mathfrak{q}}$ of H_G on $\mathfrak{q}$ and the covering map $q_{H_G} \colon \widetilde{H}_G \to H_G$.

Proposition A.3. *For a discrete central subgroup $\Gamma \subseteq \widetilde{H}_G$ acting trivially on $\mathfrak{q}$, the cosets in $\widetilde{S}$ satisfy*

$$(s\Gamma)(t\Gamma) = st\Gamma \quad \text{and} \quad (s\Gamma)^* = s^*\Gamma \quad \text{for} \quad s, t \in \widetilde{S}, \tag{33}$$

so that the quotient semigroup $\widetilde{S}/\Gamma$ inherits the structure of an analytic involutive Banach semigroup for which the quotient map $q_\Gamma \colon \widetilde{S} \to \widetilde{S}/\Gamma$ is a morphism of involutive semigroups. Moreover, the polar map $\widetilde{\Phi} \colon \widetilde{H}_G \times W \to \widetilde{S}$ factors through a diffeomorphism $\Phi_\Gamma \colon \widetilde{H}_G/\Gamma \times W \to \widetilde{S}/\Gamma$ and the group $\widetilde{H}_G/\Gamma$ acts faithfully on $\widetilde{S}/\Gamma$ by unitary multipliers.

Proof. Since the left action of Γ on $\widetilde{S}$ coincides with the right action (see (31)), the relations (33) easily follow. The remaining assertions are now obvious. $\square$

Definition A.4. (a) The semigroups obtained by the preceding proposition will be called Banach–Olshanski semigroups. We write

$$S_H(W) := \widetilde{S}_{H_G}(W)/\Gamma \quad \text{for} \quad H = \widetilde{H}_G/\Gamma$$

and $\exp x := \widetilde{\exp}(x) \cdot \Gamma$ for the exponential function $\exp\colon W \to S_H(W)$.

(b) For the special case where $\mathfrak{g} = \mathfrak{h}_{\mathbb{C}}$ and $\theta(x + iy) = x - iy$, the group G is complex, so that $\Gamma_{H_G}(W)$ is a complex manifold on which the multiplication is holomorphic and the involution is antiholomorphic. These properties are inherited by all other Olshanski semigroups $S_H(W)$, where $H \cong \widetilde{H}_G/\Gamma$ is a connected Lie group with Lie algebra $\mathfrak{h}$. Therefore we call them *complex Olshanski semigroups*.

Remark A.5. The basic properties of Olshanski semigroups $S_H(W)$ are:

(O1): The polar map $H \times W \to S_H(W), (h, x) \mapsto h \exp x$ is an analytic diffeomorphism.

(O2): H acts on $S_H(W)$ smoothly by unitary multipliers.

(O3): For $x \in W$, we have

$$\exp(sx)\exp(tx) = \exp((t + s)x) \quad \text{for} \quad t, s > 0.$$

(O4): For $h \in H$ and $s \in W$, we have $(h \exp x)^* = (\exp x)h$.

From now on H always denotes a connected Lie group with Lie algebra $\mathfrak{h}$ and $S = S_H(W)$ is a corresponding Olshanski semigroup. We do not assume that H is contained in G.

Lemma A.6. *For $s \in S_H(W)$ and $x \in W$ we have*

$$\lim_{t \to 0_+} \exp(tx)s = \lim_{t \to 0_+} s\exp(tx) = s.$$

Proof. It suffices to verify this relation in the simply connected covering semigroup $\widetilde{S}$, where we have for $s = h'\exp(x')$:

$$\exp(tx)s = \widetilde{\Phi}(e,tx)\widetilde{\Phi}(h',x') = \widetilde{\Phi}(h'\widetilde{m}_H(\mathrm{Ad}(h')^{-1}tx,x'),m_W(\mathrm{Ad}(h')^{-1}tx,x')).$$

Since the functions $\widetilde{m}_H$ and m_W extend continuously to the domain W_2, formula (30) yields $\exp(tx)s \to s$. The other relation is obtained by applying the involution $*$. $\square$

Remark A.7. Let $\kappa^r \in \Omega^1(G,\mathfrak{g})$ denote the *right Maurer–Cartan form*, defined by $\kappa^r_g(x.g) := x$ for $x \in \mathfrak{g} = T_e(G)$, where $TG \times G \to TG, (v,g) \mapsto v.g$ denotes the canonical right action of G on TG. Similarly we define the *left Maurer–Cartan form* by $\kappa^l_g(g.x) := x$.

For the subsemigroup $S = S_{H_G}(W) \subseteq G$, the restriction $\kappa^r_S := \kappa^r|_S$ defines a trivialization of the tangent bundle of S by

$$TS \to S \times \mathfrak{g}, \quad v_s \mapsto (s, \kappa^r_S(v_s)) \quad \text{for} \quad v_s \in T_s(S).$$

If $q_S \colon \widetilde{S} \to S$ is the universal covering, the form $\kappa^r_{\widetilde{S}} := q_S^*\kappa^r_S$ likewise trivializes $T(\widetilde{S})$. For every discrete central subgroup $\Gamma \subseteq \widetilde{H}_G$ acting trivially on $\mathfrak{q}$ and $S = S_{H_G}(W)$, the form $\kappa^r_{\widetilde{S}}$ on $\widetilde{S}$ is Γ-invariant, hence is the pullback of a form $\kappa^r_{\widetilde{S}/\Gamma}$ trivializing $T(\widetilde{S}/\Gamma)$.

Definition A.8. The preceding discussion shows that on every Olshanski semigroup $S_H(W)$, we have a natural form $\kappa^r_S \in \Omega^1(S,\mathfrak{g})$ trivializing the tangent bundle and we similarly obtain a left invariant form κ^l_S.

Accordingly, we have natural *left invariant vector fields* V_x, $x \in \mathfrak{g}$, on S, defined by $\kappa^l_S(V_x) = x$. and *right invariant vector fields* W_x, $x \in \mathfrak{g}$, defined by $\kappa^r_S(W_x) = x$.

Remark A.9. For $x \in \mathfrak{h} \cup W$ and $s \in S$ we have

$$V_x(s) = \frac{d}{dt}\Big|_{t=0} s\exp(tx) \quad \text{and} \quad W_x(s) = \frac{d}{dt}\Big|_{t=0} \exp(tx)s.$$

Both relations are obvious for the semigroup $S_{H_G}(W)$, and they are inherited by the simply connected covering and hence also by its quotients.

For $x \in \mathfrak{g}$ and $s \in S$, we write $t \mapsto s\exp(tx)$ for the integral curve of V_x through s and likewise $t \mapsto \exp(tx)s$ for the local integral curve of W_x. This is redundant for $S = S_{H_G}(W) \subseteq G$, and for a general $S = S_H(W)$, the preceding observation shows that it is also consistent for $x \in W \cup \mathfrak{h}$ with the action of the corresponding one-parameter (semi)groups.

Remark A.10. (a) From the right invariance of κ^r_G on G, we obtain

$$\rho_s^*\kappa^r_S = \kappa^r_S \quad \text{for} \quad s \in S = S_H(W)$$

by verifying that this property is preserved by the passage to covering semigroups and to quotients by discrete central subgroups. We likewise get

$$\lambda_s^*\kappa^l_S = \kappa^l_S \quad \text{for} \quad s \in S.$$

(b) As in (a), it follows that, for $h \in H$, the right multiplication $\rho_h \colon S \to S$ also leaves κ^r_S invariant. For the left multiplication $\lambda_h(s) = h.s$, the relation $\lambda_g^*\kappa^r_G = \mathrm{Ad}(g) \circ \kappa^r_G$ for the Maurer–Cartan form of a Lie group G implies that

$$\lambda_h^*\kappa^r_S = \mathrm{Ad}(h) \circ \kappa^r_S \quad \text{for} \quad h \in H. \tag{34}$$

This is also verified by the passage through the universal covering semigroup.

Definition A.11. For a smooth map $f\colon M \to S = S_H(W)$, where M is a smooth manifold, we define the (right) logarithmic derivative as the $\mathfrak{g}$-valued 1-form

$$\delta(f) := f^*\kappa_S^r \in \Omega^1(M, \mathfrak{g}).$$

If $I \subseteq \mathbb{R}$ is an interval and $\alpha\colon I \to S$ a differentiable path, then the identification of 1-forms on I with $\mathfrak{g}$-valued functions leads to

$$\delta(\alpha)_t := \kappa^r_{\alpha(t)}(\alpha'(t)) \in \mathfrak{g}.$$

We likewise define logarithmic derivatives for maps with values in Lie groups.

For the exponential map $\exp\colon \mathfrak{g} \to G$ we then have

$$\delta(\exp)_x(y) = \int_0^1 e^{s\,\mathrm{ad}\,x} y\, ds,$$

([Ne06, Prop. II.5.7]) and therefore:

Proposition A.12. *Any C^1-path $\alpha : I \to W$ satisfies*

$$\delta(\exp\alpha)_t = \int_0^1 e^{s\,\mathrm{ad}\,\alpha(t)}\alpha'(t) ds.$$

From the differential of the multiplications, we obtain left and right actions

$$S \times TS \to TS, \quad (s, v) \mapsto s.v, \quad TS \times S \to TS, \quad (v, s) \mapsto v.s$$

and likewise

$$H \times TS \to TS, \quad (h, v) \mapsto h.v, \quad TS \times H \to TS, \quad (v, h) \mapsto v.h$$

as well as

$$TH \times S \to TS, \quad (v, s) \mapsto v.s, \quad S \times TH \to TS, \quad (s, v) \mapsto s.v.$$

We then also have

$$(\kappa_S^r)(v.s) = v \quad \text{for} \quad v \in \mathfrak{h} = T_e(H), s \in S \tag{35}$$

because this is true for the subsemigroup $\Gamma_{H_G}(W)$ of G.

Proposition A.13. *Let $\alpha : I \to H$ and $\beta : I \to W$ be two C^1-paths. For the path $\gamma(t) := \alpha(t)\exp(\beta(t))$ in S we then have*

$$\delta(\gamma)_t = \delta(\alpha)_t + \mathrm{Ad}(\alpha(t))\delta(\exp\circ\beta)_t.$$

Proof. First we note that

$$\gamma'(t) = \alpha'(t).\exp\beta(t) + \alpha(t).(\exp\circ\beta)'(t).$$

To evaluate $\delta(\gamma)_t = (\kappa_S^r)_{\gamma(t)}(\gamma'(t))$, we first write $\alpha'(t) = \delta(\alpha)_t.\alpha(t)$ in TH. For $v \in TH$, the relation $(h'h).s = h'.(h.s)$ for $s \in S, h, h' \in H$ leads to $(v.h).s = v.(h.s)$, so that (35) implies that $\kappa_S^r(\alpha'(t).\exp\beta(t)) = \delta(\alpha)_t$. Finally, the relation (34) leads to

$$\kappa_S^r\Big(\alpha(t).(\exp\circ\beta)'(t)\Big) = \mathrm{Ad}(\alpha(t))\delta(\exp\circ\beta)_t. \qquad \square$$

Appendix B. Families of one-parameter semigroups

In this section we let $\mathfrak{g} = (\mathfrak{g}, [\cdot,\cdot])$ be a locally convex Lie algebra. This means that $\mathfrak{g}$ is a locally convex space and the Lie bracket $[\cdot,\cdot]$ is continuous. Let us first recall the following basic notions of the differential calculus over locally convex spaces.

Definition B.1. (a) Let E, F be two locally convex spaces, U open and $f : U \subseteq E \to F$ be a continuous map on the open set U of E. Then f is called C^1 if the directional derivatives $\partial_v f(x) := \lim_{h\to 0} \frac{f(x+hv)-f(x)}{h}$ exist for every $x \in U$ and $v \in E$ and the map

$$\mathtt{d}f : U \times E \to F, \quad (x, v) \mapsto \partial_v f(x)$$

is continuous.
(b) A continuous map $f : U \subseteq E \to V$ is called C^k, $k \geq 2$ if it is C^1 and $\mathtt{d}f$ is C^{k-1}. It is called C^∞, or *smooth*, if it is C^k for every $k \in \mathbb{N}$.
(c) A locally convex space E is called *Mackey complete* if for each smooth curve $\xi : [0,1] \to E$ the weak integral $\int_0^1 \xi(t)dt$ exists, i.e., there exists a (unique) element $I =: \int_0^1 \xi(t)dt \in E$ satisfying

$$\alpha(I) = \int_0^1 \alpha(\xi(t))\, dt \quad \text{for each} \quad \alpha \in E'.$$

This implies in particular that the curve $\eta(s) := \int_0^s \xi(t)dt$ is smooth and satisfies $\eta' = \xi$.

Definition B.2. A locally convex Lie algebra $\mathfrak{g}$ is called ad-*integrable* if for every $x \in \mathfrak{g}$ the (linear) vector field defined by $\operatorname{ad} x$ is complete, that is, if there exists a smooth map $\Phi^x : \mathbb{R} \times \mathfrak{g} \to \mathfrak{g}$ with $\frac{d}{dt}\Big|_{t=0} \Phi^x(t)y = \operatorname{ad} x(y)$. We will then use the notation $e^{t \operatorname{ad} x} := \Phi^x(t)$.

We consider a linear homomorphism

$$\alpha : \mathfrak{g} \to \operatorname{End}(\mathcal{D})$$

in the space of endomorphism of a dense domain $\mathcal{D}$ of a Banach space E. We assume that α is strongly continuous in the sense that for every $v \in \mathcal{D}$ the map $\alpha^v : \mathfrak{g} \to E$, $x \mapsto \alpha(x)v$ is continuous. Now let W be a convex cone in $\mathfrak{g}$ which is relatively open in its span $\mathfrak{q} := W - W$ (cf. Definition 5.1), and assume that, for every $x \in W$, the closure $\overline{\alpha(x)}$ of $\alpha(x)$ generates a strongly continuous semigroup $\left(e^{t\overline{\alpha(x)}}\right)_{t\geq 0}$ and that the map $s_\alpha(x) := \sup_{0\leq t\leq 1} \left|\left|e^{t\overline{\alpha(x)}}\right|\right|$ is locally bounded on W.

Remark B.3. If $E = \mathcal{H}$ is a Hilbert and for every $x \in W$ the semigroup $\left(e^{t\overline{\alpha(x)}}\right)_{t\geq 0}$ consists of normal operators, then the fact it generates a commutative C^*-algebra isomorphic to some $C(X)$ implies that

$$\left|\left|e^{t\overline{\alpha(x)}}\right|\right| = \left|\left|e^{\overline{\alpha(x)}}\right|\right|^t \quad \text{for } x \in W,$$

and hence that $s_\alpha(x) = \max\left\{1, \left|\left|e^{\overline{\alpha(x)}}\right|\right|\right\}$.

We will need the following lemma (see [Mer10, Lemma 9]):

Lemma B.4. *Consider two operators A and B defined on a common dense domain $\mathcal{D}$ of the Banach space E and whose closures generates strongly continuous*

semigroups $\left(e^{t\overline{A}}\right)_{t\geq 0}$ *and* $\left(e^{t\overline{B}}\right)_{t\geq 0}$ *respectively. Assume further that* $e^{sA}\mathcal{D} \subseteq \mathcal{D}$ *for all* $s \geq 0$*. If, for some* $v \in \mathcal{D}$ *the map* $s \mapsto Be^{sA}v$ *is continuous on* $[0, \infty)$*, then*

$$e^{t\overline{B}}v - e^{t\overline{A}}v = \int_0^t e^{s\overline{B}}(B - A)e^{(t-s)\overline{A}}vds \quad \text{for} \quad t \geq 0.$$

We then have the following proposition generalizing [Mer10, Prop. 10]:

Proposition B.5. *Let* $\mathfrak{g}$ *be an* ad*-integrable Mackey complete locally convex Lie algebra. If, for every* $x \in W$ *and every* $y \in \mathfrak{g}$*,*

$$e^{\overline{\alpha(x)}}\mathcal{D} \subseteq \mathcal{D} \quad \text{and} \quad \alpha(y)e^{\overline{\alpha(x)}} = e^{\overline{\alpha(x)}}\alpha(e^{-\operatorname{ad} x}y), \tag{36}$$

then the map

$$\widehat{\rho} \colon W \times E \to E, \ (x, v) \mapsto \rho^v(x) := e^{\overline{\alpha(x)}}v$$

is continuous, and for every $v \in \mathcal{D}$*,* ρ^v *is* C^1 *with*

$$T_x(\rho^v)(y) = \alpha\Big(\int_0^1 e^{s\operatorname{ad} x}yds\Big)e^{\overline{\alpha(x)}}v = e^{\overline{\alpha(x)}}\alpha\Big(\int_0^1 e^{-s\operatorname{ad} x}yds\Big)v.$$

Proof. Let $v \in \mathcal{D}$ and $x, y \in \mathfrak{g}$. From the relation

$$\alpha(y)e^{s\overline{\alpha(x)}}v = e^{s\overline{\alpha(x)}}\alpha(e^{-s\operatorname{ad} x}y)v, \tag{37}$$

the continuity of the map $[0, \infty[\times E \to E, (s, v) \mapsto e^{s\overline{\alpha(x)}}v$ and the strong continuity of α, we derive that $s \mapsto \alpha(y)e^{s\overline{\alpha(x)}}v$ is continuous on $[0, \infty[$. We can therefore apply Lemma B.4 to obtain

$$\begin{aligned} e^{\overline{\alpha(y)}}v - e^{\overline{\alpha(x)}}v &= \int_0^1 e^{u\overline{\alpha(y)}}\alpha(y - x)e^{(1-u)\overline{\alpha(x)}}vdu \\ &= \int_0^1 e^{u\overline{\alpha(y)}}e^{(1-u)\overline{\alpha(x)}}\alpha\big(e^{(u-1)\operatorname{ad} x}(y - x)\big)vdu. \end{aligned} \tag{38}$$

Let $\varepsilon > 0$ let $\mathcal{V}$ be a 0-neighbourhood in $\mathfrak{g}$ such that for every $y \in x+\mathcal{V}$, $s_\alpha(y) < M$. Consider now the continuous map

$$F : [0, 1] \times \mathfrak{g} \to E, \ (u, z) \mapsto \alpha^v\big(e^{(u-1)\operatorname{ad} x}z\big).$$

Since $F([0, 1] \times \{0\}) = \{0\}$, the compactness of $[0, 1]$ implies the existence of a 0-neighborhood $\mathcal{V}' \subseteq \mathfrak{g}$ such that $F([0, 1] \times \mathcal{V}')$ is contained in the open ball $B(0, \frac{M}{\varepsilon})$ of radius $\frac{M}{\varepsilon}$ around 0 in E. Thus, for every $y \in x + \mathcal{V} \cap \mathcal{V}'$, $\left|e^{\overline{\alpha(y)}}v - e^{\overline{\alpha(x)}}v\right| \leq \varepsilon$, and this proves that ρ^v is continuous. The local boundedness of $s_\alpha \colon W \to \mathbb{R}$ further implies that ρ^v is continuous for every $v \in E$, and hence the map

$$\widehat{\rho} \colon W \times E \to E, \ (x, v) \mapsto e^{\overline{\alpha(x)}}v$$

is continuous.

Let $y \in \mathfrak{q}$ and let $\tau > 0$ such that $x + hy \in W$ for $|h| < \tau$. We derive from (38) the formula

$$\frac{e^{\overline{\alpha(x+hy)}}v - e^{\overline{\alpha(x)}}v}{h} = \int_0^1 e^{s\overline{\alpha(x+hy)}}\alpha(y)e^{(1-s)\overline{\alpha(x)}}vds.$$

Let us fix $0 < \varepsilon \leq 1$. Then the continuity of the map

$$(s, h) \mapsto e^{s\overline{\alpha(x+hy)}}\alpha(y)e^{(1-s)\overline{\alpha(x)}}v$$

on $[\varepsilon, 1] \times [-\tau, \tau]$ implies that we can pass to the limit under the integral sign to derive

$$\lim_{h\to 0}\int_\varepsilon^1 e^{s\overline{\alpha(x+hy)}}\alpha(y)e^{(1-s)\overline{\alpha(x)}}vds = \int_\varepsilon^1 e^{s\overline{\alpha(x)}}\alpha(y)e^{(1-s)\overline{\alpha(x)}}vds. \tag{39}$$

The same type of argument as the one used for the continuity of ρ^v shows that the integrand of

$$\int_0^\varepsilon e^{s\overline{\alpha(x+hy)}}\alpha(y)e^{(1-s)\overline{\alpha(x)}}vds$$

is bounded unformly with respect to (s,h), and hence the integral is uniformly small (with respect to h) when ε is sufficently close to 0. Therefore

$$\begin{aligned}(\partial_y\rho^v)(x) &= \int_0^1 e^{s\overline{\alpha(x)}}\alpha(y)e^{(1-s)\overline{\alpha(x)}}vds = \int_0^1 \alpha(e^{s\,\mathrm{ad}\,x}y)e^{\overline{\alpha(x)}}vds \\ &= \alpha\Big(\int_0^1 e^{s\,\mathrm{ad}\,x}yds\Big)e^{\overline{\alpha(x)}}v,\end{aligned}$$

where the last equality follows from the uniqueness of the weak integral. Similarly we obtain

$$\partial_y\rho^v(x) = e^{\overline{\alpha(x)}}\alpha\Big(\int_0^1 e^{-s\,\mathrm{ad}\,x}yds\Big)v,$$

and now the continuity of $\widehat{\rho}$ implies that $T\rho^v : W\times\mathfrak{q}\to E$ is continuous, i.e, that ρ^v is a C^1-map. $\square$

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